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A SIMULATION STUDY OF A CLASS OF FIRST-COME FIRST-SERVED QUEUES--ETC(U)

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A Simulation Study of a Class of First-Come First-Served Queues with EARMA Correlation Structure

by

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Submitted in partial fulfillment of the requirements for the degree of

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ABSTRACT

The object of this thesis is to examine, via simulation, the properties of a class of single-server, first-come first-served queues in which the service times and interarrival times, while still marginally exponential, are auto-correlated and cross-correlated. The correlation is introduced by letting the service and interarrival sequences be EARMA-type processes, where EARMA stands for exponential autoregressive, moving average sequence. An extension of these ideas brings in the cross-correlation. The waiting times in the dependent queue are compared to the waiting times (with known distribution) in the independent M/M/1 queue using data analytic and formal statistical methods. Variance reduction techniques are also studied; these use distributionally known aspects of the M/M/1 queue (simulated with the same exponential sequences as the dependent queue) to control the unknown aspects of the dependent queue.

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I. INTRODUCTION

The object of this thesis is to examine, via simulation, the properties of a class of single-server first-come first-served queues in which the service times and interarrival times, while still marginally exponential, are auto-correlated and cross-correlated. The correlation is obtained by making the service and interarrival time sequences multivariate EARMA processes. An ancillary aspect of the study is the investigation of the efficiency of a number of control variable variance reduction schemes. These schemes are made possible by the simplicity of the structure of the multivariate EARMA processes, and the fact that the null case (no correlation) gives the M/M/1 queue, whose properties are known analytically.

An immediate problem which arises in the simulation is that it is not known how far out along the sample path the simulation must be carried in order for the steady state properties of the queue to hold. This is a general problem which arises in queuing simulations and, in order to cope with it, the simulations are run in 500 parallel replications. This allows one to use many methods of time series analysis and data analysis to obtain graphical verification in a sequential manner of the approach to a steady state. The replications also allow for examination of the complete steady state waiting time distribution in the queue, and not merely the mean waiting time.

While all of these aspects of the queuing simulation are interwoven, we can separate them out for purposes of discussion as follows.

(i) An elementary aspect of the queue which can be examined is the mean waiting time. The most efficient (smallest variance) estimate of this is the sample path average which is again averaged over replications. This replication of course reduces the variance by a factor of m, where m is the number of replications. Furthermore the existence of replications allows one to look at the complete distribution of the sample path average estimate. This distribution should be normally distributed, possibly even before a steady state has been reached.

The object of this part of the study is to see how the mean steady state waiting time in the queue varies with the traffic intensity, t, and the correlation parameters, and in particular, how this mean waiting time differs from the waiting time in the M/M/1 queue (null case).

(ii) A more detailed aspect of the queue which has to be examined is the complete distribution of the steady state waiting time. In the M/M/1 queue this is exponential, given that the waiting time is greater than zero.

The first problem here is to determine that one is actually looking at the steady state waiting time. This is done by examining the empirical distribution

function of the waiting times across replications at times n_1 and n_2 ($n_2 > n_1$), where the index n refers to the customer number. Of course the two samples from W_{n_1} and W_{n_2}, each of size m, will be correlated and this makes standard statistical methods for comparing samples invalid. However the correlation between these samples can be calculated in order to determine that n_2 is sufficiently large relative to n_1 for the correlation to be negligible. In actual fact the comparison is done by looking at box plots of the W_n's at successive values $n_1 < n_2 < n_3 < \ldots$. This method is graphic and allows the simulator to see how the steady state is being approached.

(iii) A third problem which is involved in looking at the complete distribution of the waiting time is that a sample of size m = 500 is not sufficient to really get a good estimate of the steady state distribution of the waiting time. Thus successive samples along the parallel sample paths are combined, where it is determined that the samples are far enough apart to be approximately independent.

Again the object here is to see how the complete distribution of the steady state waiting time differs from the exponential distribution (null M/M/1 case) as traffic intensity, t, and correlation parameters vary. In particular heavy traffic theory says that this distribution should be approximately exponential

as the traffic intensity parameter approaches 1. The difficulty in verifying this, and the heavy traffic approximation to the mean waiting time, is that the time to reach the steady state becomes very long as $t \to 1$. An additional question that arises then is whether it is possible to start the simulation in such a way (approximate steady state conditions) so as to speed up this convergence. It should be remembered, that the queue with correlation is not regenerative and no initial conditions for stationarity are known or are likely to be found.

(iv) Variance reduction is always helpful in a simulation of this kind since real detail is always masked by sampling fluctuations i.e., the variability in the estimate of the unknown mean waiting time. Because of the simple probabilistic-linear structure of the EARMA processes, running on M/M/1 queue in parallel with the correlated queue using common exponential variates provides a very good control variable for the waiting time if the correlation, in a rough sense, in the EARMA queue, is small. However, as this correlation increases, the amount of control decreases. Thus several other control variables are examined and combined into a multiple control for the correlated queue. In this way it is hoped to obtain variance reduction for any values of the correlation parameters

in the EARMA queue. The actual degree of control which can be attained is examined, using the replicated simulation, as a function of the parameters of the queue.

II. THE EXPONENTIAL AUTOREGRESSIVE-MOVING AVERAGE PROCESS, EARMA(1,1)

The first order moving average process and the first-order autoregressive processes were combined by Jacobs and Lewis (1977) to form the EARMA(1,1) sequence. We first describe these two first order sequences, and then the method of combining them.

A. THE EXPONENTIAL AUTOREGRESSIVE PROCESS, EAR(1)

The standard linear, first-order autoregressive model for a stationary sequence of random variables $\{X_i\}$ is defined by the equation

$$X_i = \rho X_{i-1} + \varepsilon_i \; ; \quad i = 0, \pm 1, \pm 2, \ldots , \tag{II.1}$$

where ρ is a constant which is less than 1 in absolute value and $\{\varepsilon_i\}$ is a sequence of independent and identically distributed random variables. Gaver and Lewis (1978) showed that if the $\{X_i\}$ sequence were to have an exponential marginal distribution with parameter λ, then the parameter ρ should be greater than or equal to zero and less than one, and ε_i should be zero with probability ρ or an exponential (λ) random variables, E_i, with probability $1-\rho$. Thus

$$X_i = \rho X_{i-1} + \varepsilon_i \; ; \quad i = 0, \pm 1, \pm 2, \ldots ,$$

becomes

$$
(II.2) \quad X_i = \begin{cases} \rho X_{i-1} & \text{with probability } \rho \\ \rho X_{i-1} + E_1 & \text{with probability } 1-\rho, \end{cases}
$$

$i = 0, \pm 1, \pm 2, \ldots$ where $\{E_i\}$ is an identical independent distribution (i.i.d.) sequence of exponential (λ) random variables. Note that for this EAR(1) model the distribution of the ε_i depend on ρ, the multiplicative weight of X_{i-1}. Also ε_i is not an absolutely continuous random variable. Thus standard results (Mallows, 1968) to the effect that the X_i in an autoregressive process become approximately normal as $\rho \to 1$ do not hold. In fact in the EAR(1) process (II.2), the X_i are exponentially distributed for $0 \leq \rho < 1$.

B. THE EXPONENTIAL MOVING AVERAGE PROCESS, EMA(1)

The first order moving average exponential process EMA(1) was developed by Lawrance and Lewis (1977) in the form of

$$
(II.3) \quad X_i = \begin{cases} \beta E_i & \text{w.p. } \beta \\ \beta E_i + E_{i-1} & \text{w.p. } (1-\beta), \end{cases}
$$

$i = 0, \pm 1, \pm 2, \ldots$, where β is greater than or equal to zero and less than or equal to one, and $\{E_i\}$ is again a sequence of i.i.d. exponential (λ) random variables. (This is the backward case — a forward case is also possible.) The X_i's have an exponential marginal distribution and are only serially

dependent for lag one; this model is highly tractable and a full account of the statistically useful properties was obtained by Lawrance and Lewis (1977). The forward model combines E_i with E_{i+1} instead of with E_{i-1}.

C. THE EXPONENTIAL AUTOREGRESSIVE-MOVING AVERAGE PROCESSES EARMA(1,1)

The EAR(1) model of Section IIA and the EMA(1) model of Section IIB are combined in the following form to give an EARMA(1,1) process. This is a non-Markovian model which contains the EMA(1) and EAR(1) models as a special case. The defining equations for the EARMA(1,1) process are

$$\text{(II.4)} \qquad X_i = \begin{cases} \beta E_i, & \text{w.p. } \beta, \\ \beta E_i + A_{i-1}, & \text{w.p. } 1-\beta, \end{cases}$$

$$(0 \le \beta \le 1;\ i = 0, \pm 1, \pm 2, \ldots)$$

where

$$A_{i-1} = \begin{cases} \rho A_{i-2} & \text{w.p. } \rho, \\ \rho A_{i-2} + E_{i-1}, & \text{w.p. } 1-\rho, \end{cases}$$

$$(0 \le \rho \le 1;\ i = 0, \pm 1, \pm 2, \ldots)$$

and $\{E_i\}$ is, as usual, a sequence of i.i.d. exponential (λ) random variables. Thus instead of E_i being combined with the exponential random variable E_{i-1}, it is combined with an

exponentially distributed random variable which is an EAR(1) combination of E_{i-1}, E_{i-2}, E_{i-3}, ... The serial correlations for the EARMA(1,1) process are given in Jacobs and Lewis (1977) as

$$\rho(j) = \operatorname{corr}(X_i X_{i+j}) = c(\rho,\beta)\rho^{j-1}$$

$$(j = 0, \pm 1, \pm 2, \pm 3, \ldots;\ 0 \leq \rho < 1;\ 0 \leq \beta \leq 1).$$

where $c(\rho,\beta) = \beta(1-\beta)(1-\rho) + (1-\beta)^2\rho$.

Note that when $\rho = 0$ the EARMA(1,1) process reduces to an EMA(1) process; if $\beta = 0$ it is the autoregressive EAR(1) process, and when $\beta = 1$ it is the usual sequence of exponential (λ) random variables. We will say that X_i is autoregressive over E_i, E_{i-1}, ...

III. USE OF MIXED STRUCTURES EARMA(1,1) IN MODELLING QUEUES

Consider for simplicity a queue with a single input stream and a single server and a first-come-first-served (FIFO) service discipline. Let S_i, $i = 0, 1, 2, \ldots$, denote the service time for the ith arrival, and let X_i, $i = 1, 2, \ldots$, denote times between arrival of the ith and (i-1)th customers (interarrival times). As is usual we assume that the first customer (with service time S_0) arrives at time zero and finds the queue empty.

If the $\{S_i\}$ and $\{X_i\}$ sequences are i.i.d. exponential random variables with parameters λ and α respectively, we have M/M/1 queue.

Now let

E_i be i.i.d. exponential (λ); $i = 0, 1, 2, \ldots$,

ε_i be i.i.d. exponential (α); $i = 0, 1, 2, \ldots$.

We want to model queues with correlated (autocorrelated and/or cross-correlated) service and inter-arrival times, the service and inter-arrival times both having marginally exponential distribution (MDxMD/1 queue). Exponential marginals is a common assumption and with it the MDxMD/1 queuing model includes the M/M/1 queue as a special case.

The dependence in the arrival and service processes is created here by defining S_i and X_i as a bivariate dependent sequence of random variables with exponential marginal distributions. This is done by letting $\{S_i\}$ be EARMA(1,1) over E_i, $(\alpha/\lambda)\varepsilon_i$, $(\alpha/\lambda)\varepsilon_{i-1}$, ... , i.e.

$$S_0 = E_0$$

$$S_1 = \begin{cases} \beta E_1 & \text{w.p. } \beta; \\ \beta E_1 + A_1, & \text{w.p. } (1-\beta); \end{cases} \qquad A_1 = (\alpha/\lambda)\varepsilon_1;$$

$$S_2 = \begin{cases} \beta E_2 & \text{w.p. } \beta; \\ \beta E_2 + A_2, & \text{w.p. } (1-\beta); \end{cases} \qquad A_2 = \begin{cases} \rho A_1 & \text{w.p. } \rho; \\ \rho A_1 + \frac{\alpha}{\lambda}\varepsilon_2 & \text{w.p. } (1-\rho); \end{cases}$$

$$\vdots$$

$$S_i = \begin{cases} \beta E_i, & \text{w.p. } \beta; \\ \beta E_i + A_i, & \text{w.p. } (1-\beta); \end{cases} \qquad A_i = \begin{cases} \rho A_{i-1} & \text{w.p. } \rho; \\ \rho A_{i-1} + \frac{\alpha}{\lambda}\varepsilon_i & \text{w.p. } (1-\rho); \end{cases}$$

$$\vdots$$

We also let

$$X_i = \varepsilon_i; \quad i = 1, 2, 3, \ldots ,$$

although one could further make the X_i's correlated. With the present assumptions the input process is still a Poisson process, as in the M/M/1 queue. The cross-coupling between the sequence $\{S_i\}$ and $\{X_i\}$ is apparent from the structure. Note that the S_i's are now correlated because of the cross-coupling. The $\{S_i\}$ is a sequence of dependent exponential random variables, but not an EARMA(1,1) sequence. It is a type of pseudo-EARMA(1,1) sequence.

The interpretation of this queuing model is that the server tends to speed up if the queue gets long (in the past). Of course he also slows down when the queue gets short. In the case where $\rho = 0$ and $\beta = 0$ then the correlation between $\{S_i\}$ and $\{X_i\}$ will be equal to one, i.e., ρ $S_i = (\alpha/\lambda)X_i$. Also when $\beta = 1$, and for any value of ρ, the queuing model will be identical to the M/M/1 queue. Thus the M/M/1 is included as a special case. Some more complicated schemes for coupling the service and inter-arrival processes are given in Lewis and Shedler (1978). Jacobs (1978a, 1978b) has studied heavy traffic approximations for EARMA-type queues.

Other attempts to model M/M/1 (FIFO) type queues with correlated service and interarrival times have been made. None of them, however, have given fixed (exponential) marginal distributions for S_i and X_i, or cross-coupling between these sequences. There are, of course, Markovian schemes based on birth-death representations which give FIFO queues in which the service times and interarrival times are correlated

(Cox and Smith, 1961), but the scheme is quite different from that given here. In particular the interarrival and service times are not exponentially distributed, and this may not conform to what is observed in practice.

IV. THE SIMULATION MODEL

A. INTRODUCTION AND NOTATION

Since both queues we are considering, the M/M/1 queue and the MDxMD/1, are single-server, first-in-first out queues, the waiting times W_n in both are given by the recursion equation

$$W_o = 0$$

$$W_{n+1} = \max(W_n + S_n - X_{n+1}, 0) \; ; \quad n = 0,1,2, \ldots$$

These equations are used to generate successive W_n's in the simulation. This is the only aspect of the queue which will be considered. To have looked at quantities like the number of customers in the queue at departure times would have complicated the programming and taken away from the primary aim of the thesis, which is to explore the effect of correlation on the queue and to use some relatively modern statistical methodology to do this.

We denote the waiting time of the nth customer to arrive in the jth realization of the queue by $W_n(j)$. The usual sample path estimate for the mean, $E(w)$, of the limiting distribution of waiting times, $F_w(x)$, where

$$F_w(x) = \lim_{n\to\infty} P\{W_n \leq x\} ,$$

is the sample path average

$$\overline{W}_n(j) = \frac{1}{n} \sum_{\ell=0}^{n} W_\ell(j)$$

which can be shown to converge to the mean E(W) as $n \to \infty$.

The M independent sample path realizations $W_n(j)$, $n = 0,1,2, \ldots$, $j = 1,2, \ldots, M$ can be used to obtain estimates of

(1) the distribution of the estimate $\overline{W}_n(j)$. This should be normal for large n. This can be examined from the sample $\overline{W}_n(j)$, $j = 1, \ldots, M$. In addition the mean of this sample

$$\overline{\overline{W}}_n = \frac{1}{M} \sum_{j=1}^{M} \overline{W}_n(j) = \frac{1}{Mn} \sum_{j=1}^{M} \sum_{\ell=0}^{n} W_\ell(j)$$

is a grand mean estimating E(w) whose variance is computable from the sample $\overline{W}_\ell(j)$, $j = 1, \ldots, M$.

Thus

$$\mathrm{Var}(\overline{\overline{W}}_\ell) = \frac{1}{M}(\text{sample variance of } \overline{W}_\ell(j)\text{'s})$$

$$= \frac{1}{M}(\frac{1}{M-1} \sum_{j=1}^{M} (W_n(j) - \overline{\overline{W}}_n)^2)$$

(2) The distribution of W_n (not only its mean) from the sample $W_n(j)$, $j = 1, \ldots, M$. Since this is a random sample, all the classical methods are available, e.g., histograms, normal plots, empirical c.d.f's.

(3) The correlations between successive waiting times, such as $\mathrm{corr}(W_{n_1}(j), W_{n_2}(j))$, $n_1 < n_2$. This is given as

$$\frac{1}{M}\left(\sum_{j=1}^{M} (W_{n_1}(j) - \hat{W}_{n_1})(W_{n_2}(j) - \hat{W}_{n_2})\right),$$

divided by the sample standard deviations of the two samples, where

$$\hat{W}_{n_1} = \frac{1}{M} \sum_{j=1}^{M} (W_{n_1}(j))$$

is an average across replications.

B. PROGRAM STRUCTURE

1. General

To simulate this particular EARMA(1,1) with cross-correlated service time and inter-arrival time, a FORTRAN program has been written to calculate the mean waiting time $W_n(j)$. It also calculates the mean of various statistics for the M/M/1 queue using the same exponential deviates that are used to generate the correlated queue. These statistics

are used to control the estimates from the MDxMD/1 queue for purposes of variance reduction. The maximum number of customers which the program can handle for each replication is n = 10,000. This computation is divided into a maximum of 10 steps and the samples of waiting times at these points are stored in a file. Thus for example, if we use the maximum size n = 10,000, we have ten data sets $W_{1000}(j)$, $W_{2000}(j)$, ..., $W_{10,000}(j)$, j = 1, ..., M. Thus the time evolution of the queue can be studied. Moreover there is a facility to restart the simulation to get $W_{11,000}(j)$, $W_{12,000}(j)$ ... This way the evolution of the queuing process can be studied as far out as needed, e.g., to n = 270,000, or n = 1,000,000 etc. Another program will take care of reading from the output file to do data analysis, i.e., box plot of waiting times at various steps, average waiting times $\overline{W}_N(j)$ and their statistics.

2. <u>MAIN1 Program</u>

a) The variables are to be read in as follows:

N	=	Number of arrivals to be generated (max is 10,000)
RS, RS	=	Arrival, Service rate.
BETA, RHOS	=	β, ρ
KN	=	Number of the run
KS	=	1 if first run, so that n = 0,1,...,N
		0 otherwise (i.e., restart)
NREP	=	Number of replications (max 500)
NSTEP	=	Number of steps in this run (maximum of 10)

NPAST	=	Number of arrivals of last run (i.e., last run stopped at n = 120,000)
IEX, IES	=	Seeds to generate exponential random variable of Arrival, Service times
IBRTA, IRHOS	=	Seeds to generate uniform random variables to generate probabilities in the EARM(1,1) process of moving average and auto-regressive.

b) The output will be stored in two separated files.

i) For continuation of the generation of the waiting times of each replication to the next run (i.e., the restart), the variables stored are as follow:

Correlated queue

ARIV	=	cumulative inter-arrival time (these are the same for both queues)
SERV	=	cumulative service time
SNEW	=	service time of the last customer in this run
WAIT	=	waiting time of the last customer in this run
WBAR	=	cumulative waiting time
ALAST	=	Auto-regressive component of the inter-arrival time of the last customer in this run
TL	=	Number of customers who have arrived since the last zero waiting time, i.e., backward recurrence time in the renewal process of times of emptiness in the queue
TR	=	Number of zero waiting times up to the termination point of this run.

Uncorrelated queue

The variables are defined as the same as for the correlated queue; in the program the quantities are

suffixed by M, e.g. WAITM is the waiting time for the uncorrelated queue and WAIT the waiting time for the correlated queue.

ii) For analyses and displays of the data in each run and each step the data, including the data with control variables added, are stored in another file in the sequence of waiting time in each replication of the correlated queue, average waiting time in each replication of the correlated queue, waiting time in each replication of the uncorrelated queue, average waiting time in each replication of M/M/1 queue, waiting time (controlled) and average waiting time (controlled) of the correlated queue.

3. <u>Subroutine CARMA</u>

This subroutine is used to create a string of service times, S_n, which are cross-correlated to the inter-arrival time X_n, and also to create the moving average structure.

4. <u>Subroutine CONVAR, COPYSM, COPYMV and MPROCV</u>

These four subroutines are used to compute the statistics from the M/M/1 queue which are used to control the data from MDxMD/1 queue for purposes of variance reduction.

5. <u>MAIN2 Program with Subroutine COMPAR, FILL and STAT</u>

These programs and subroutines are used to display and analyze the data for each run. They produce the figures given in this thesis. (See Appendix II.)

a. Description of Box Plot

Subroutine COMPAR will produce vertical box plots (McNeil 1977) parallel to each other and write out the minimum and the maximum value of all the data in the array.

Each box plot is obtained by first calculating the lower and upper quartiles and the median of the batch of numbers.

Where the <u>median</u> of a batch of numbers is the value for which half of the numbers in the batch are larger and half are smaller, the <u>lower quartile</u> is the value that divides the batch into two parts, with 1/4 of the numbers below this value, and 3/4 above it. Similarly the <u>upper quartile</u> is the value for which 3/4 of the numbers are below it and 1/4 above it.

The narrow rectangular box with ends corresponding to the lower and upper quartiles contains the median point (*). The height of the box, the <u>interquartile distance</u>, D , is then measured out on each side of the quartiles. Then the lowest and highest data points that fall between these quantities are also marked by crosses (X). Finally, any numbers whose positions are outside these crosses are marked with circles (O), those more than 1.5 interquartile distances outside each quartile getting the symbol (@). The digit besides the "outlier" positions is to indicate the number of data points that are too close to be marked separately. (See Figure 1 .)

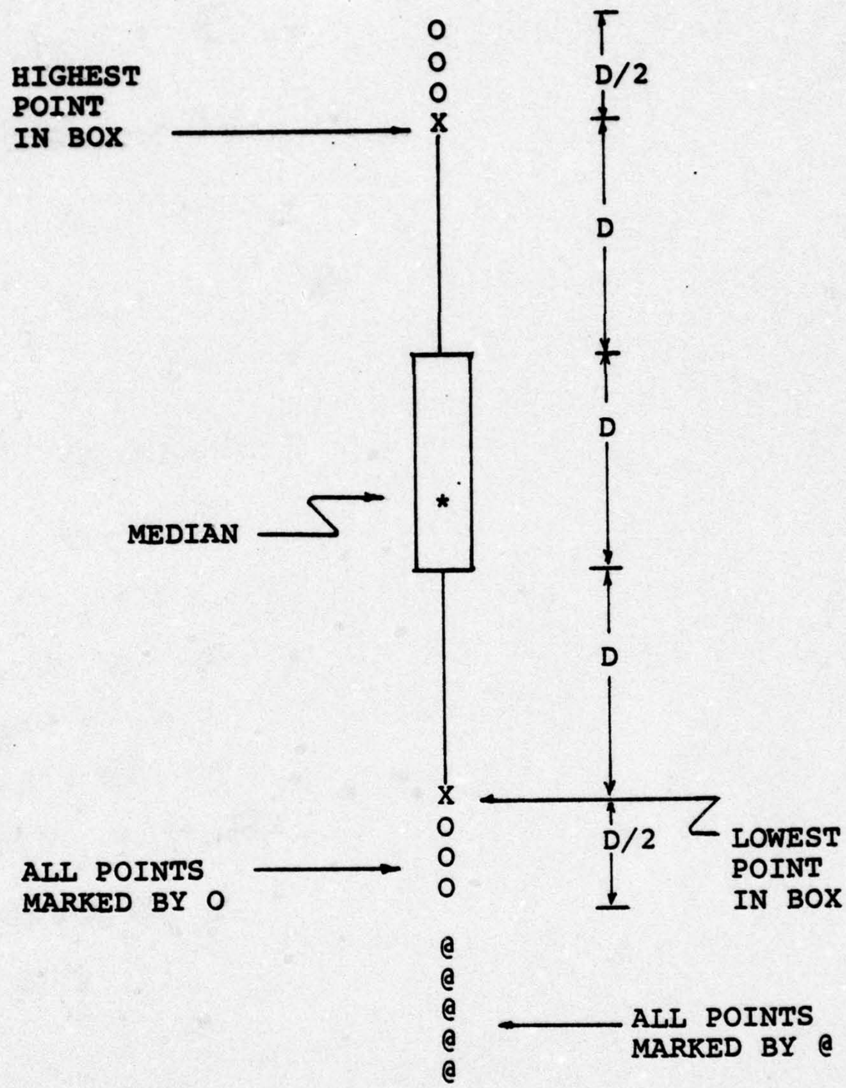


FIGURE 1. Description of Box Plot

V. ANALYSIS OF SIMULATION RESULTS

A. INTRODUCTION

The object of the simulation was to examine the effect of *the cross correlation*, indexed by the coefficients ρ and β in the EARMA(1,1) process, on the stationary waiting time W for various values of the traffic intensity t. Both the mean and distribution of W were compared to the known mean and distribution of the waiting time in an M/M/1 queue (same t and same mean service time).

To perform this comparison we initially simulated this model for values .25, .50, .75, .90 of the taffic intensity (t), for values 0.0, .25, .50, .75, & .90 of the autoregressive parameter (ρ), and the single value 0.50 for the moving average coefficient (β) for each combination of t and ρ. These values were chosen to limit the amount of computing to a physically feasible range, with the idea that higher values of ρ and t would be examined if time permitted. The number of customers, n, until steady state is reached goes up dramatically as ρ and t increase. The simulations are run in 500 parallel replications, and the 10,000 arrivals for each run are divided into 10 steps of 1,000 arrivals.

Successive groups of 10,000 arrivals were run for each combination of ρ and t until it was determined from an analysis of the output that the simulation had reached a steady state. This determination was made from box-plots of samples

$W_{n_1}(j)$, $W_{n_2}(j)$, $j = 1, \ldots, M$, for $n_2 > n_1$, from plots of $\hat{W}_n$ and $\bar{W}_n$, and from formal statistical comparisons of the two samples with n_2 large enough compared to n_1 that the correlation between the two samples is negligible.

B. ESTIMATION OF TIME TO STEADY STATE

For each run, the distribution of the sample path average estimate $\bar{W}_n(j)$ for each step is observed by the method of box plots. By comparing these plots along the n axis at steps of arrivals of 1,000, convergence to steady-state and normality can be determined. Figures 1a-g show the convergence to steady state of the M/M/1 queue at $t = .75$. For the EARMA-type MDxMD/1 queue with $t = .75$, $\rho = .75$, $\beta = .50$, the box plots are shown in Fig. 2a-g. The number of arrivals at the estimated steady state for the M/M/1 queue and for the correlated queue for any combination of ρ and t are shown in table 1.

In Figures 1 and 2 note that the variance of $\bar{W}_n(j)$ is decreasing as $1/n$, so that there is a continual shrinkage from left to right in the plots. The symmetry and lack of extreme values gives an indication of normality of the estimates. Note that the scales on successive figures change through 1a to 1g etc. This is because the plots are not commensurate. Range of values is indicated by the values at top and bottom on the left, e.g., 2.3460522 and 0.3639930 in Figure 1a.

Traffic intensity, t

Correlation, ρ	.25	.50	.75	.90
.00	40,000	40,000	50,000	50,000
.25	40,000	50,000	50,000	70,000
.50	40,000	50,000	50,000	70,000
.75	40,000	50,000	70,000	100,000
.90	60,000	70,000	100,000	100,000

Table 1. Number of arrivals that the simulation run indicates are needed approximately to reach steady state (for $\beta=0.5$). These are read off of plots such as those given for $t = 0.75$, $\rho = 0.75$ and $\beta = 0.5$ in Figures 2a-g.

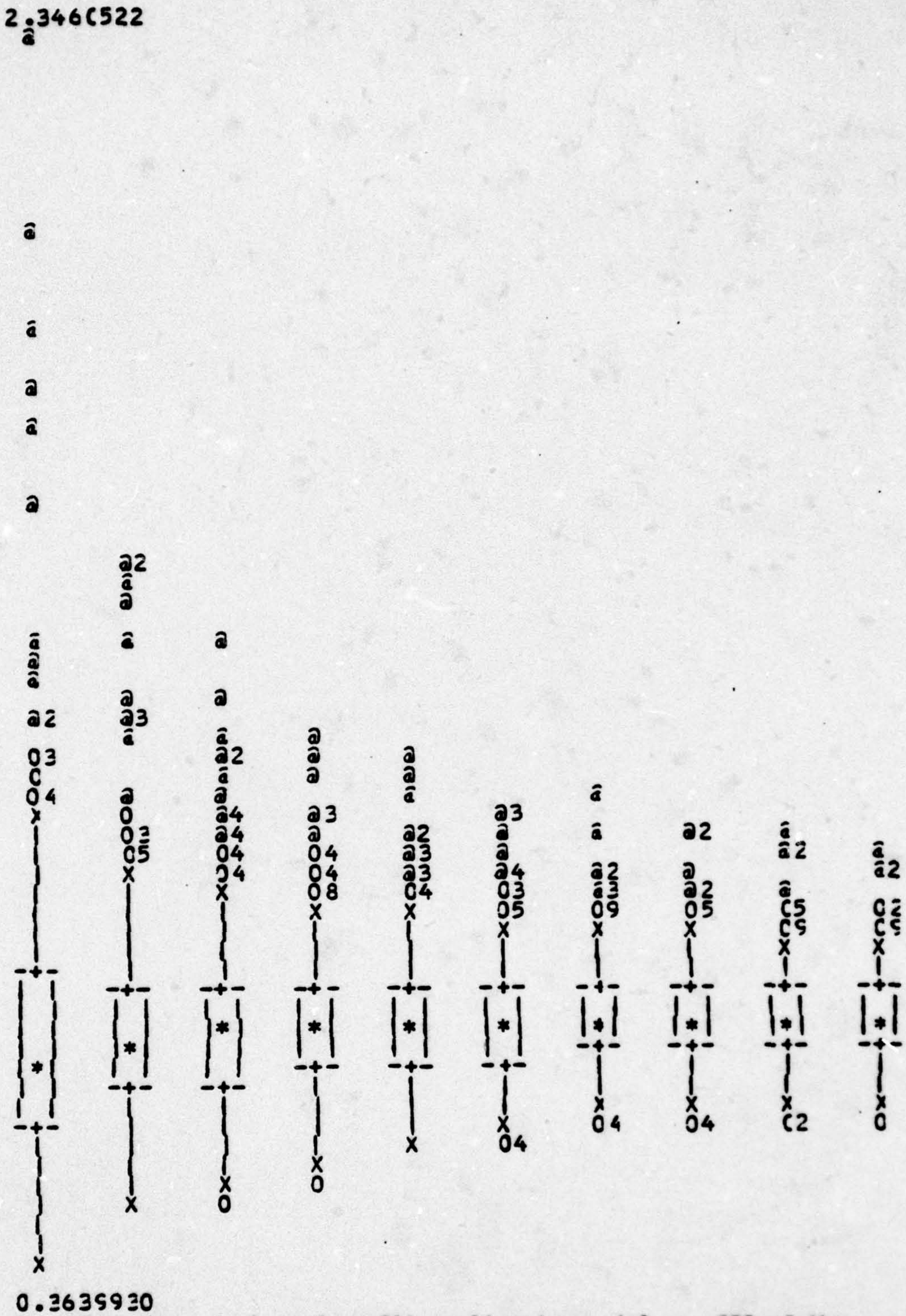


Figure 1a. Box plots derived from 500 replications, j=1,...,500 of the sample path estimate $\overline{W}_n(j)$ of the mean waiting time. The mean of these, $\overline{W}_n$, is an overall estimate of the mean waiting time. The medians of the $\overline{W}_n(j)$ are given by the * in the box plot.

Uncorrelated queue (M/M/1)
N = 1,000 to 10,000 in steps of 1,000
t = 0.75, 1/E(S) = Service rate = 4.0

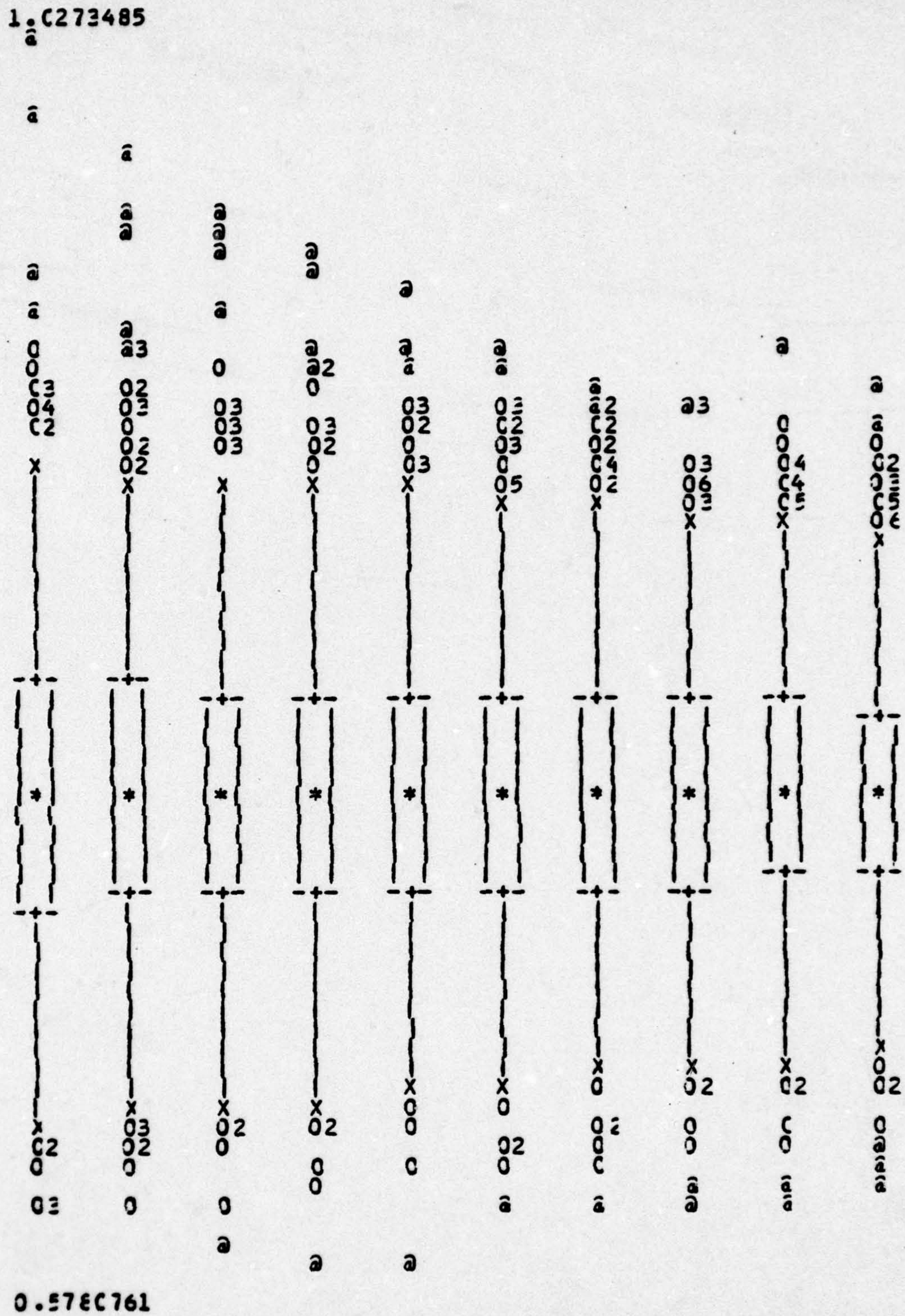


Figure 1b. Box plots derived from 500 replications, j=1,...,500 of the sample path estimate $\overline{W}_n(j)$ of the mean waiting time. The mean of these, $\overline{W}_n$, is an overall estimate of the mean waiting time. The medians of the $\overline{W}n(j)$ are given by the * in the box plot.
<u>Uncorrelated queue</u> (M/M/1)
N = 11,000 to 20,000 in steps of 1,000
t = 0.75, 1/E(S) = Service rate = 4.0

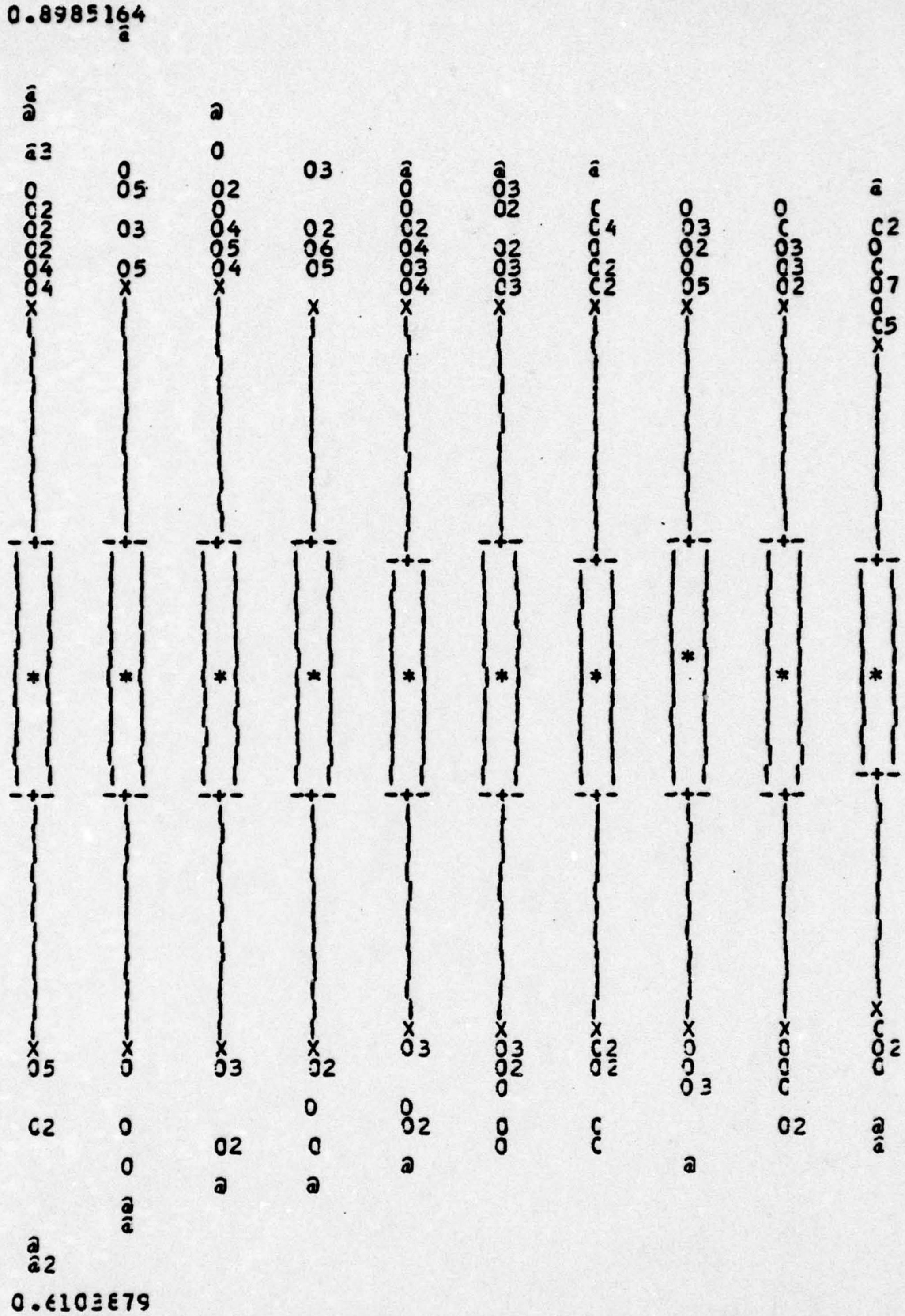


Figure 1c. Box plots derived from 500 replications, j=1,...,500 of the sample path estimate $\overline{W}_n(j)$ of the mean waiting time. The mean of these, $\overline{\overline{W}}_n$, is an overall estimate of the mean waiting time. The medians of the $\overline{W}_n(j)$ are given by the * in the box plot.
<u>Uncorrelated queue</u> (M/M/1)
N = 21,000 to 30,000 in steps of 1,000
t = 0.75, 1/E(S) = Service rate = 4.0

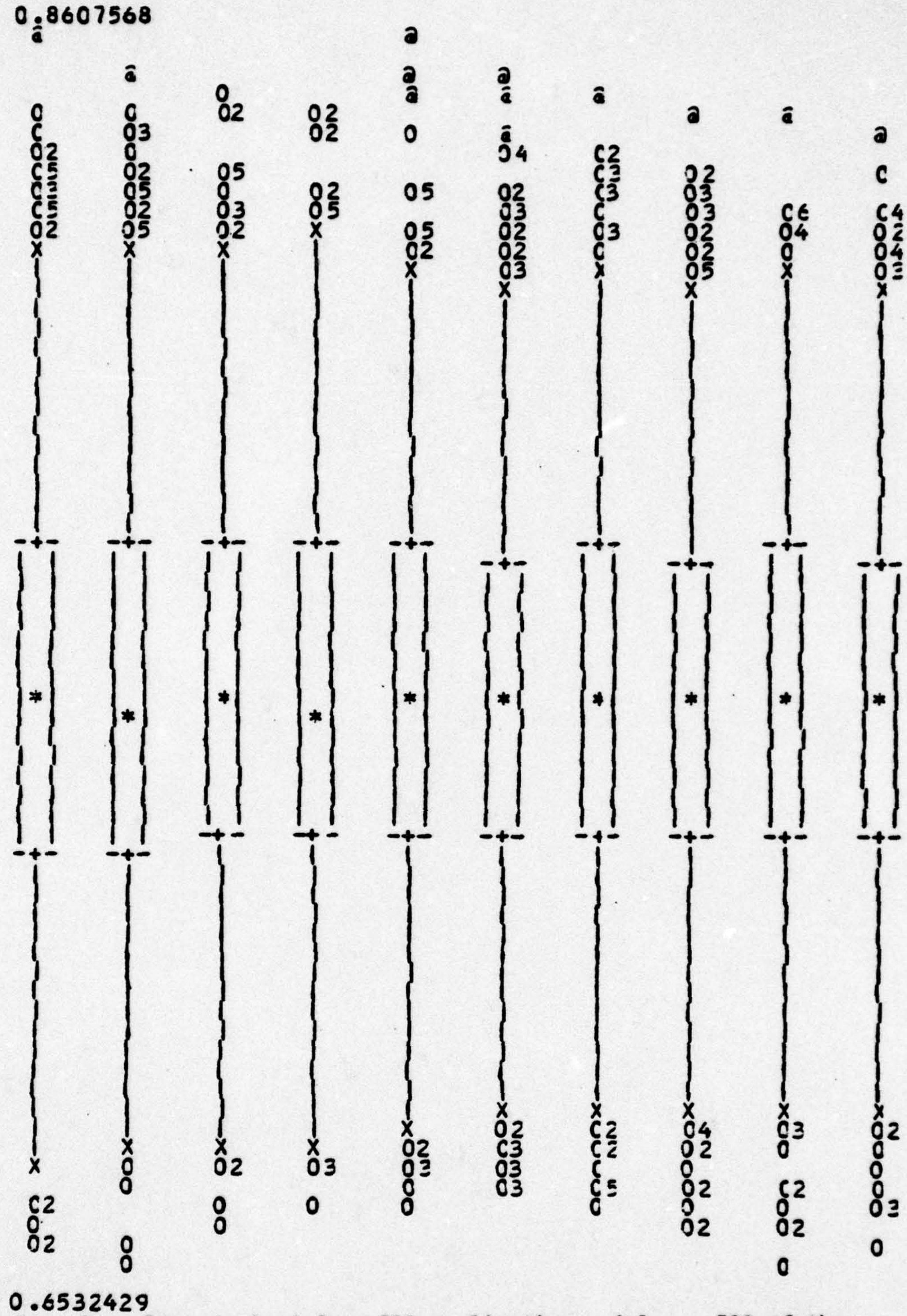


Figure 1d. Box plots derived from 500 replications, j=1,...,500 of the sample path estimate $\overline{W}_n(j)$ of the mean waiting time. The mean of these, $\overline{\overline{W}}_n$, is an overall estimate of the mean waiting time. The medians of the $\overline{W}_n(j)$ are given by the * in the box plot.
Uncorrelated queue (M/M/1)
N = 31,000 to 40,000 in steps of 1,000
t = 0.75, 1/E(S) = Service rate = 4.0

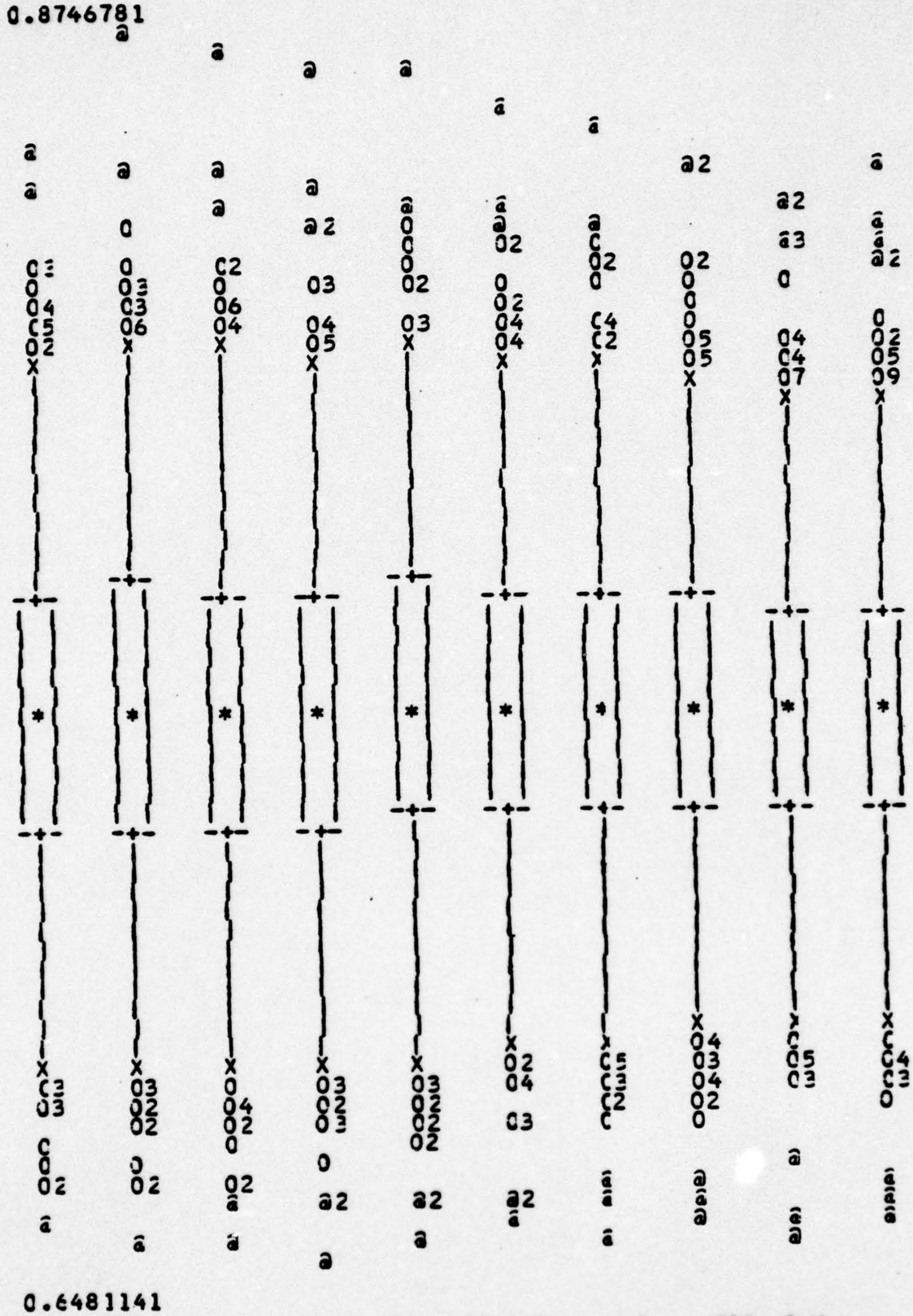


Figure 1e. Box plots derived from 500 replications, j=1,...,500 of the sample path estimate $\overline{W}_n(j)$ of the mean waiting time. The mean of these, $\overline{\overline{W}}_n$, is an overall estimate of the mean waiting time. The medians of the $\overline{W}_n(j)$ are given by the * in the box plot.
<u>Uncorrelated</u> <u>queue</u> (M/M/1)
<u>N = 41,000 to 50,000 in steps of 1,000</u>
t = 0.75, 1/E(S) = Service rate = 4.0

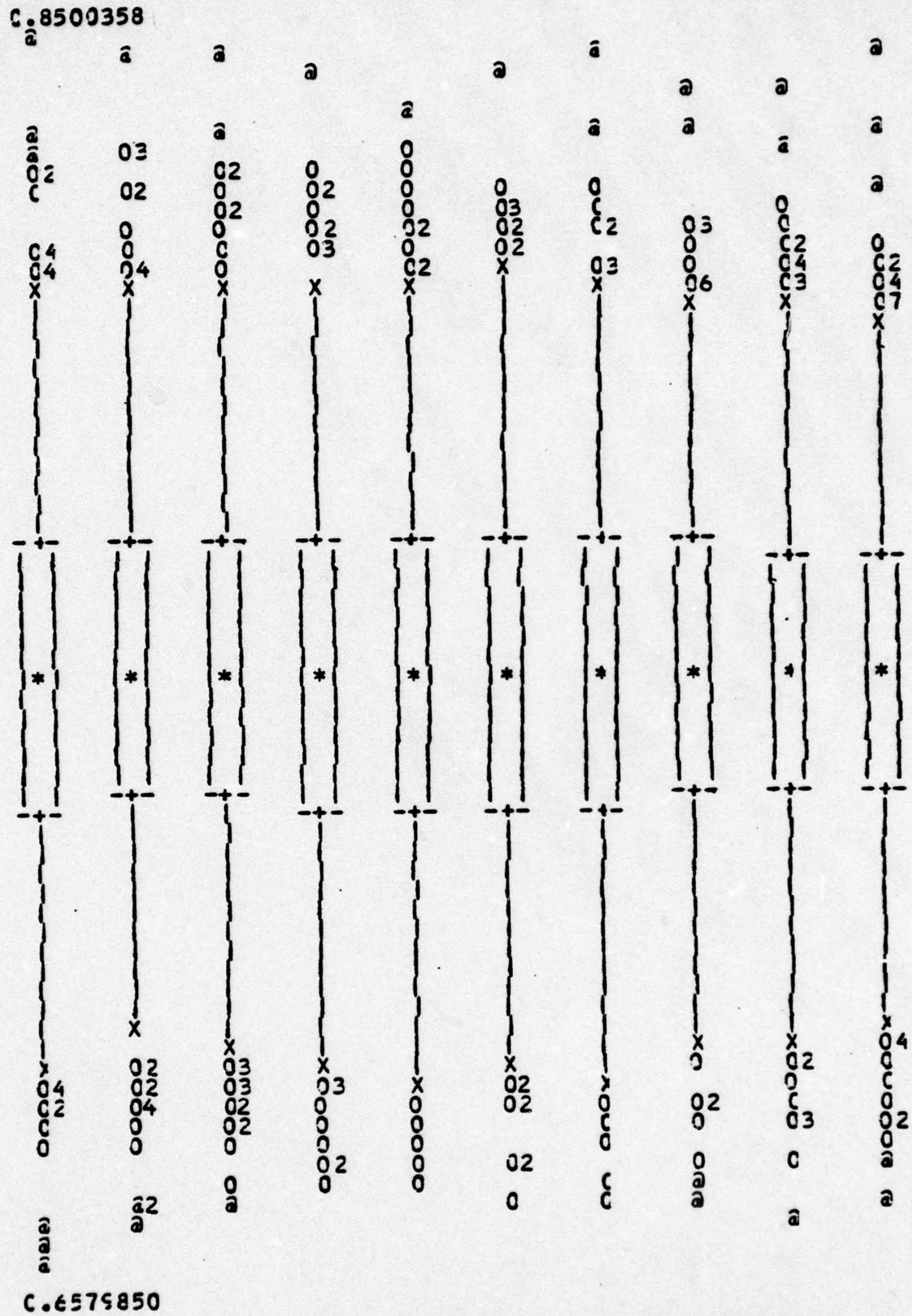


Figure 1f. Box plots derived from 500 replications, j=1,...,500 of the sample path estimate $\overline{W}_n(j)$ of the mean waiting time. The mean of these, $\overline{\overline{W}}_n$, is an overall estimate of the mean waiting time. The medians of the $\overline{W}_n(j)$ are given by the * in the box plot.
Uncorrelated queue (M/M/1)
N = 51,000 to 60,000 in steps of 1,000
t = 0.75, 1/E(S) = Service rate = 4.0

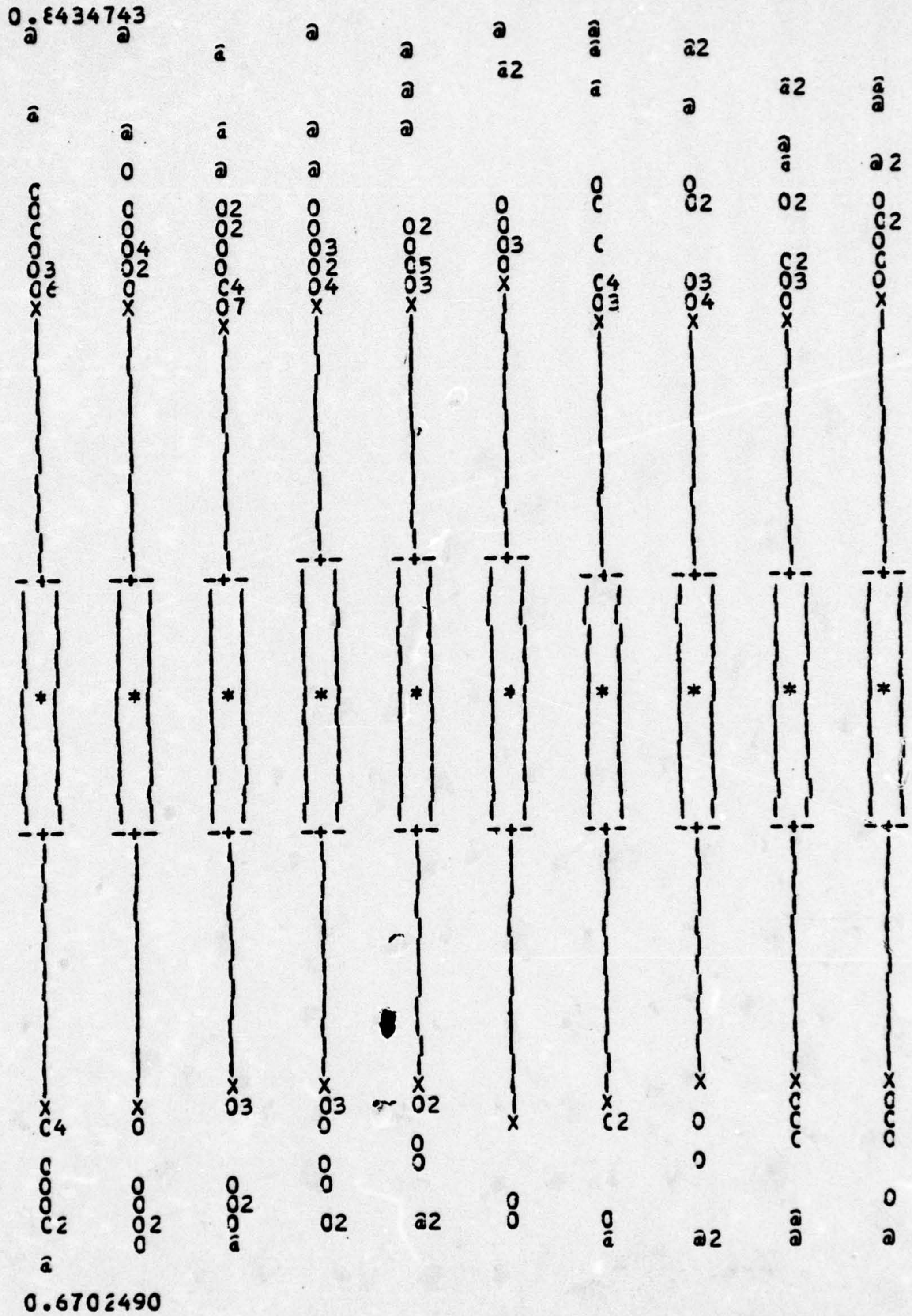


Figure 1g. Box plots derived from 500 replications, j=1,...,500 of the sample path estimate $\overline{W}_n(j)$ of the mean waiting time. The mean of these, $\overline{W}_n$, is an overall estimate of the mean waiting time. The medians of the $\overline{W}_n(j)$ are given by the * in the box plot.
<u>Uncorrelated queue</u> (M/M/1)
<u>N = 61,000 to 70,000</u> in steps of 1,000
t = 0.75, 1/E(S) = Service rate = 4.0

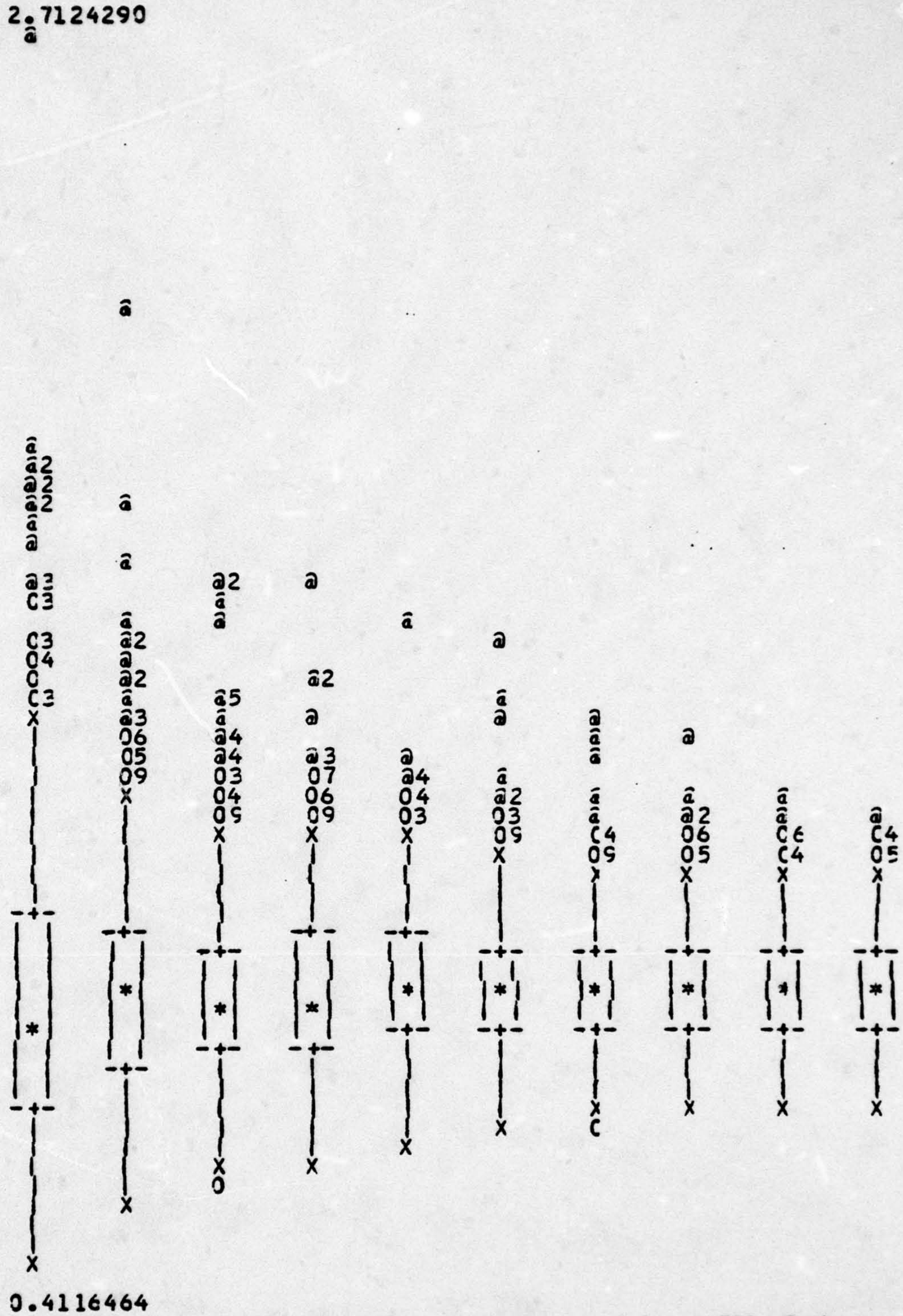


Figure 2a. Box plots derived from 500 replications, j=1,...,500 of the sample path estimate $\overline{W}_n(j)$ of the mean waiting time. The mean of these, $\overline{W}_n$, is an overall estimate of the mean waiting time. The medians of the $\overline{W}_n(j)$ are given by the * in the box plots.
<u>Correlated queue</u> (MDxMD/1)
N = 1,000 to 10,000 in steps of 1,000
t = 0.75, β = 0.5, ρ = 0.75. Service rate = 4.0

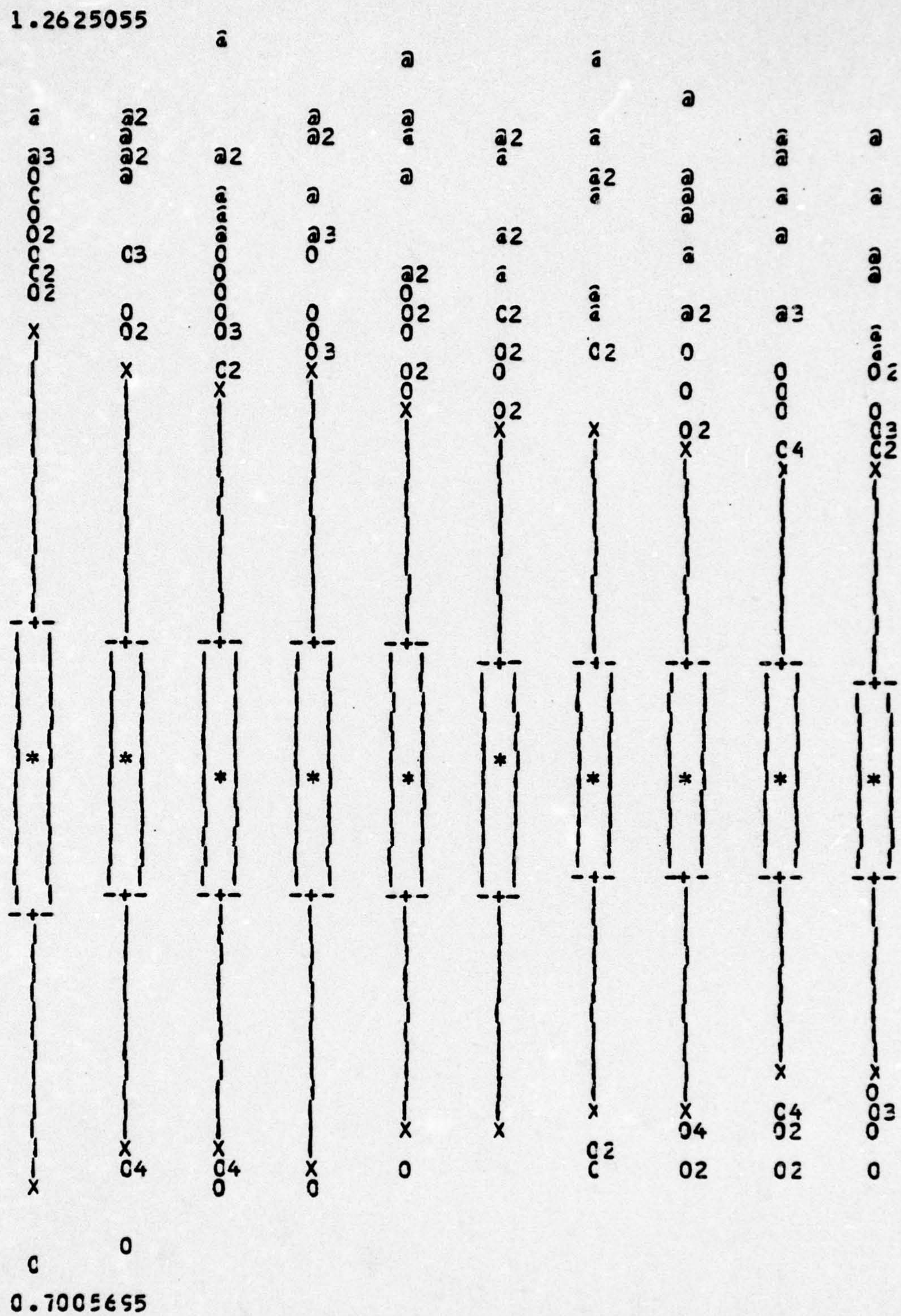


Figure 2b. Box plots derived from 500 replications, j=1,...,500 of the sample path estimate $\overline{W}_n(j)$ of the mean waiting time. The mean of these, $\overline{W}_n$, is an overall estimate of the mean waiting time. The medians of the $\overline{W}_n(j)$ are given by the * in the box plots.
<u>Correlated queue</u> (MD×MD/1)
N = 11,000 to 20,000 in steps of 1,000
t = 0.75, β = 0.5, ρ = 0.75. Service rate = 4.0

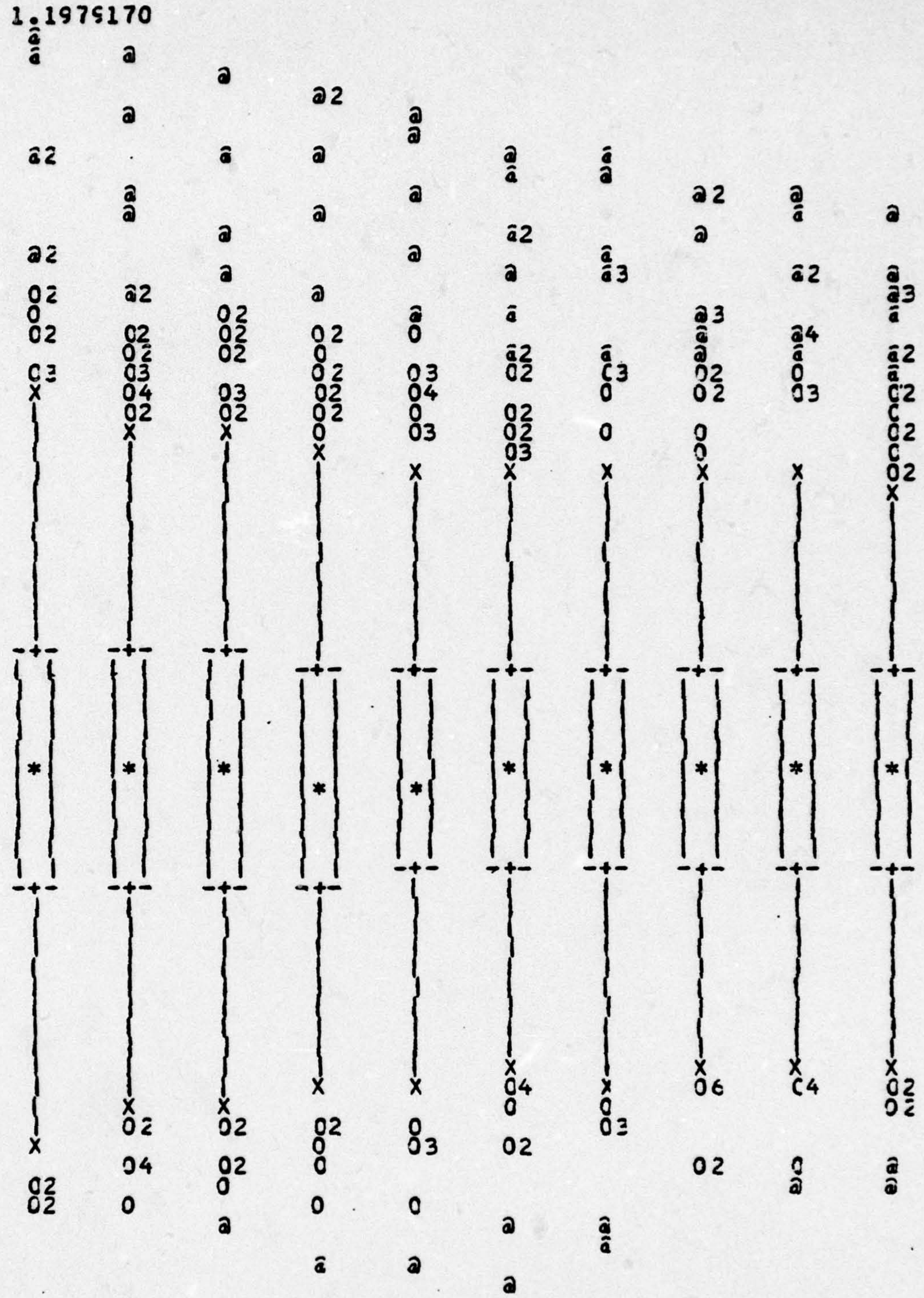


Figure 2c. Box plots derived from 500 replications, j=1,...,500 of the sample path estimate $\overline{W}_n(j)$ of the mean waiting time. The mean of these, $\overline{W}_n$, is an overall estimate of the mean waiting time. The medians of the $\overline{W}_n(j)$ are given by the * in the box plots.

<u>Correlated</u> <u>queue</u> (MD×MD/1)

$N = 21{,}000$ to $30{,}000$ in steps of 1,000

$t = 0.75$, $\beta = 0.5$, $\rho = 0.75$. Service rate = 4.0

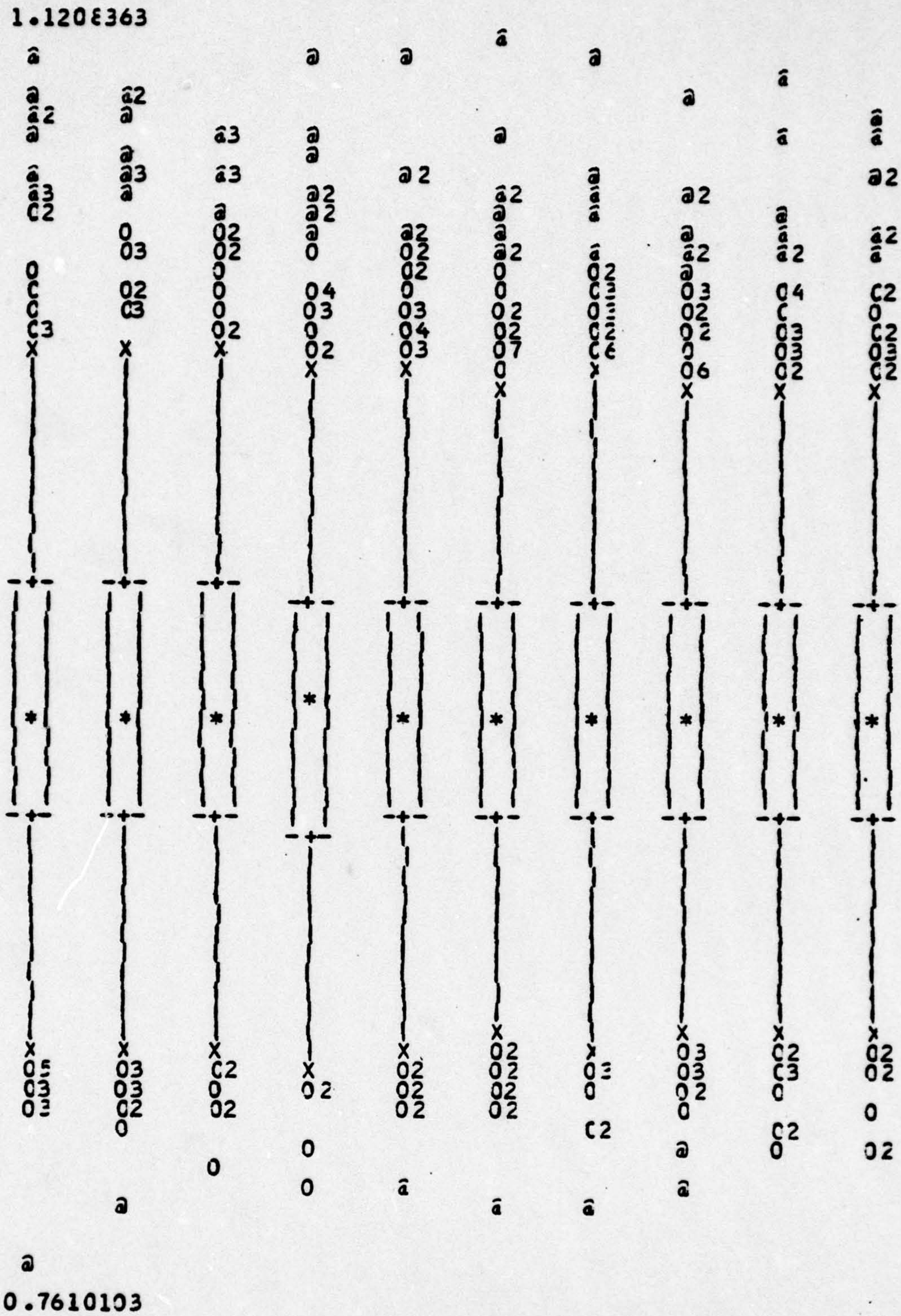


Figure 2d. Box plots derived from 500 replications, j=1,...,500 of the sample path estimate $\overline{W}_n(j)$ of the mean waiting time. The mean of these, $\overline{W}_n$, is an overall estimate of the mean waiting time. The medians of the $\overline{W}_n(j)$ are given by the * in the box plots.
Correlated queue (MD×MD/1)
N = 31,000 to 40,000 in steps of 1,000
$t = 0.75$, $\beta = 0.5$, $\rho = 0.75$. Service rate = 4.0

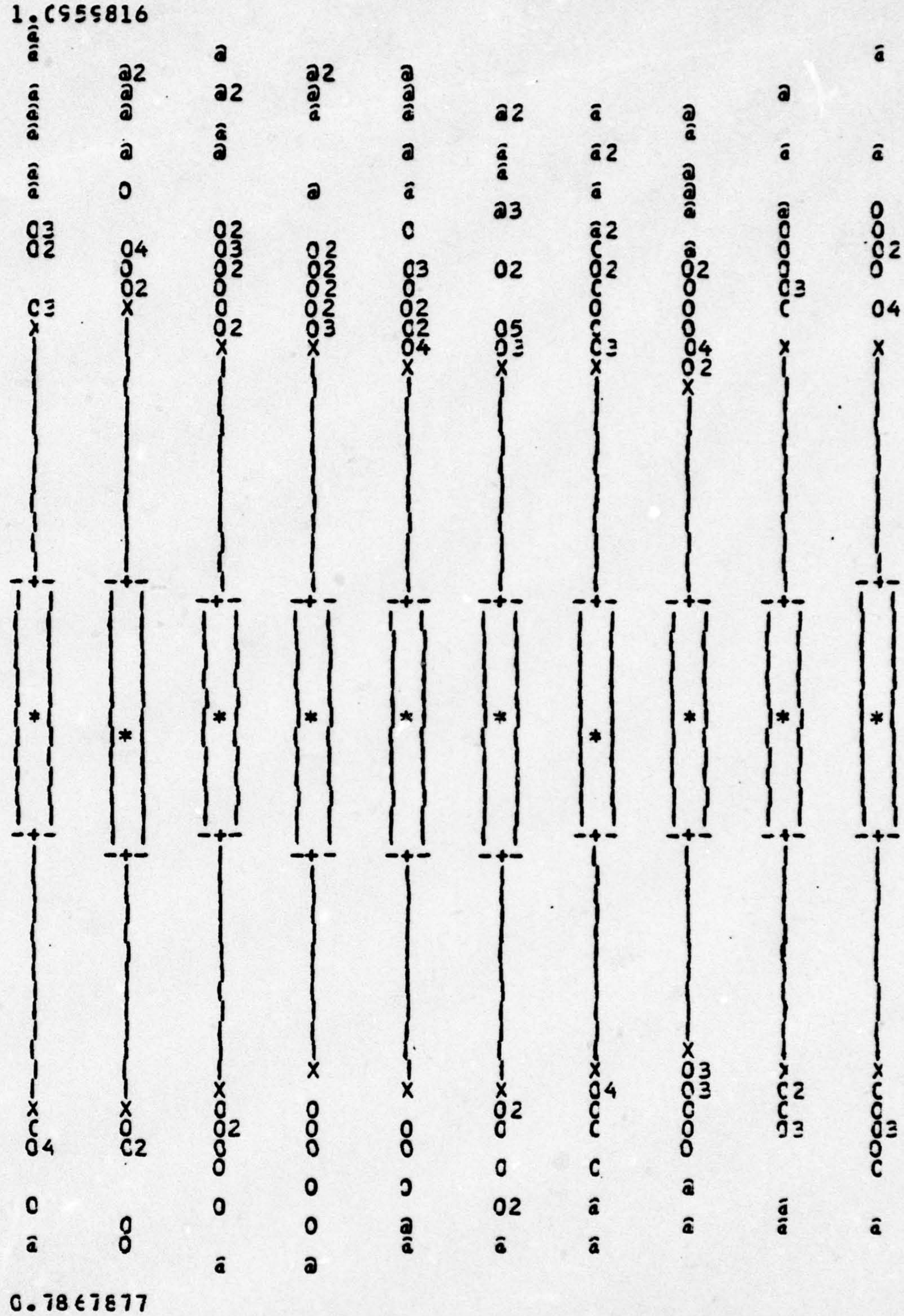


Figure 2e. Box plots derived from 500 replications, j=1,...,500 of the sample path estimate $\overline{W}_n(j)$ of the mean waiting time. The mean of these, $\overline{\overline{W}}_n$, is an overall estimate of the mean waiting time. The medians of the $\overline{W}_n(j)$ are given by the * in the box plots.
<u>Correlated queue</u> (MD×MD/1)
<u>N = 41,000 to 50,000</u> in steps of 1,000
$t = 0.75$, $\beta = 0.5$, $\rho = 0.75$. Service rate = 4.0

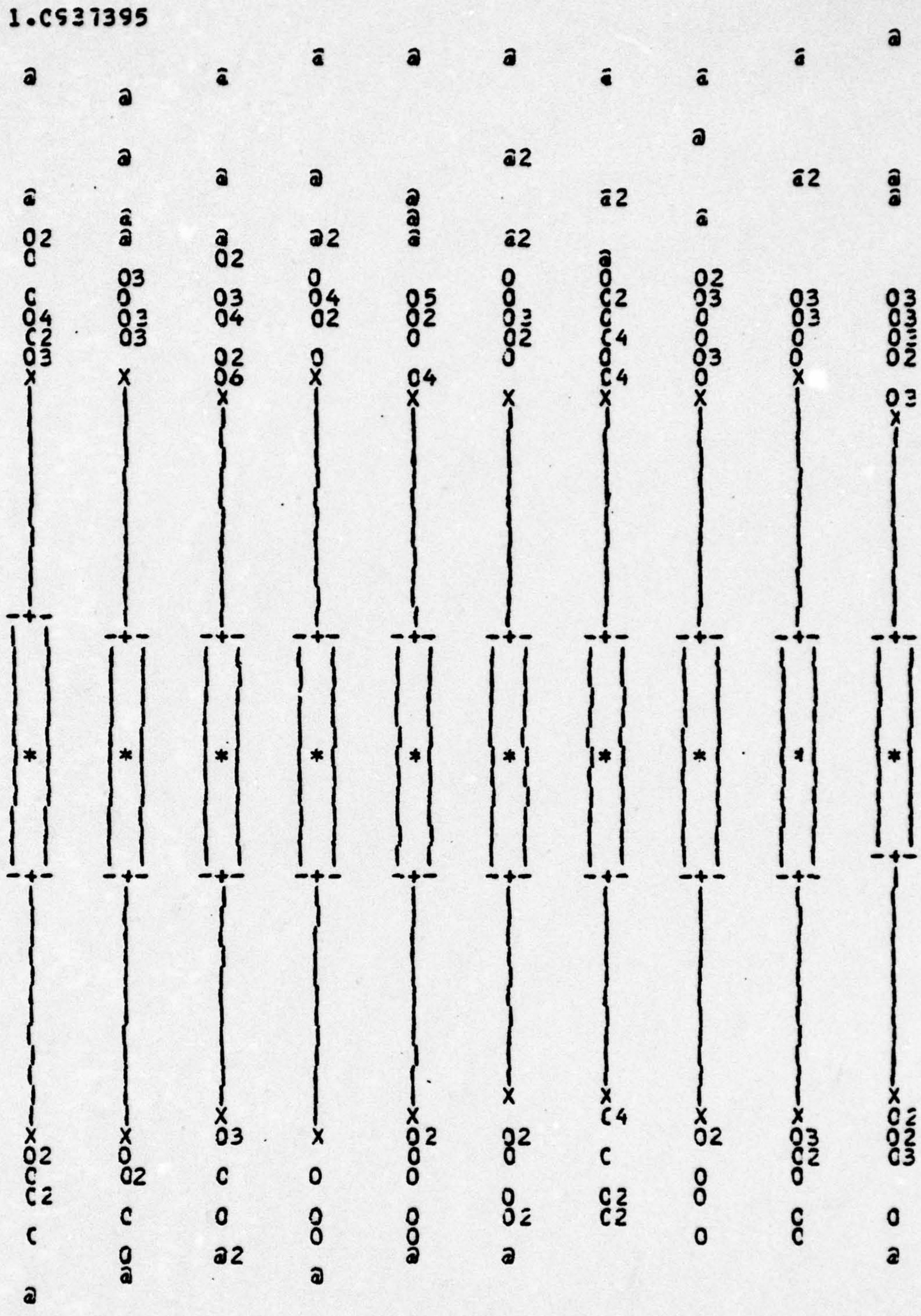


Figure 2f. Box plots derived from 500 replications, j=1,...,500 of the sample path estimate $\overline{W}_n(j)$ of the mean waiting time. The mean of these, $\overline{\overline{W}}_n$, is an overall estimate of the mean waiting time. The medians of the $\overline{W}_n(j)$ are given by the * in the box plots.
<u>Correlated</u> <u>queue</u> (MD*MD/1)
<u>N = 51,000 to 60,000</u> in steps of 1,000
$t = 0.75$, $\beta = 0.5$, $\rho = 0.75$. Service rate = 4.0

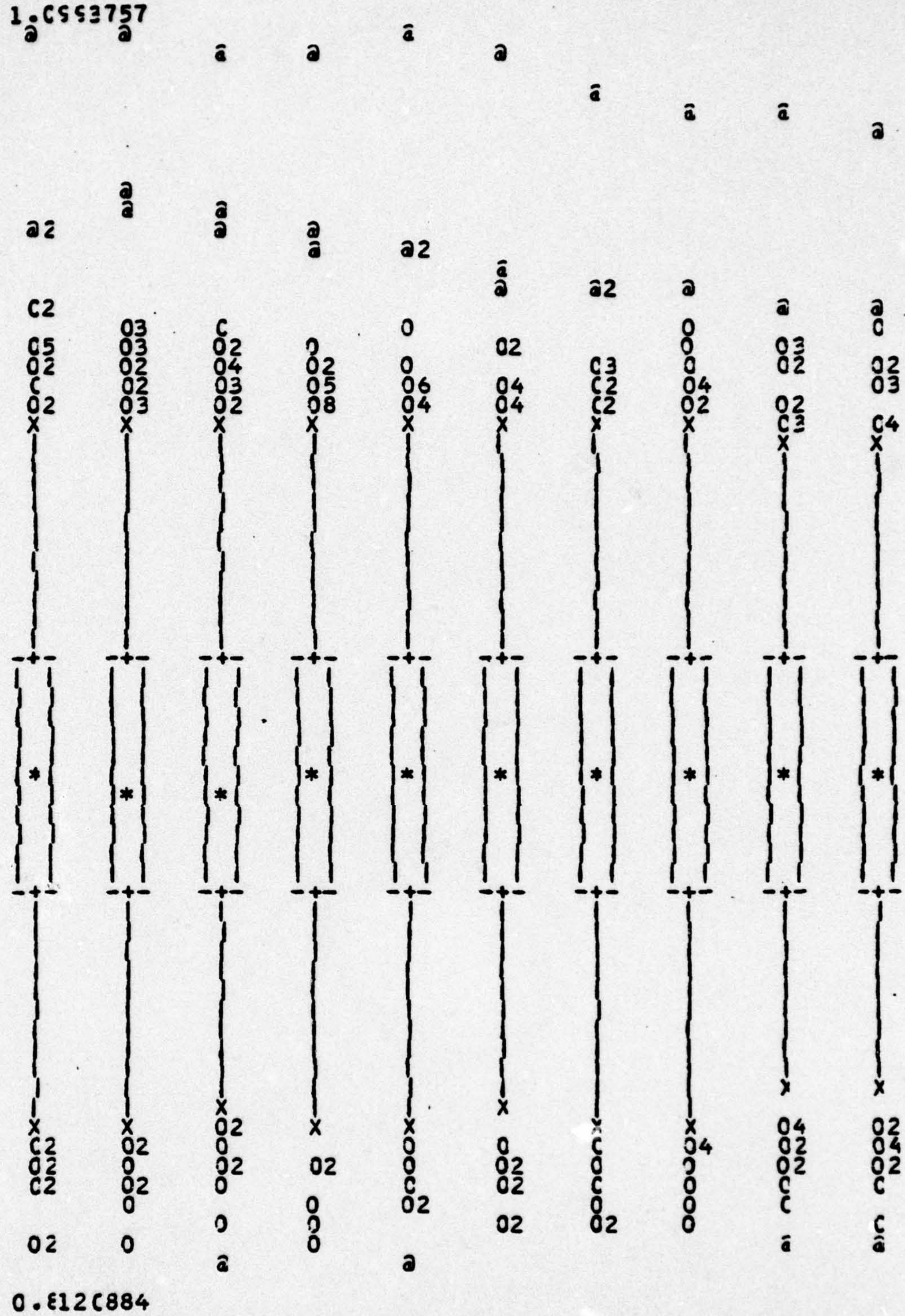


Figure 2g. Box plots derived from 500 replications, j=1,...,500 of the sample path estimate $\overline{W}_n(j)$ of the mean waiting time. The mean of these, $\overline{W}_n$, is an overall estimate of the mean waiting time. The medians of the $\overline{W}_n(j)$ are given by the * in the box plots.
<u>Correlated queue</u> (MD*MD/1)
$N = 61,000$ to $70,000$ in steps of 1,000
$t = 0.75$, $\beta = 0.5$, $\rho = 0.75$. Service rate = 4.0

C. ESTIMATED MEAN WAITING TIMES

To compare the change of the estimate steady state mean waiting time from the M/M/1 queue as ρ is changed, Table 2a shows the value of mean waiting time, $\bar{\bar{W}}_n$ and the estimate standard deviation of those means. The mean service rate is fixed at 4.0.

The mean waiting times of the correlated queue for any fixed value of ρ and t, are plotted in Figures 3a and 3b respectively. These show that the mean waiting time increases dramatically as $t \to 1$, just as for the M/M/1 queue ($\beta = 1.0$), for which the mean waiting time increases as $1/(1-t)$. A similar drastic increase in the mean waiting time because of the increase in cross-correlation, measured by ρ is apparent in Figure 3b, when t is large. Note that when ρ is close to one there is very long term correlation in the queuing process; however, $\rho = 1$ is not allowed because the system is not ergodic.

The ratio of the estimated mean waiting time, $\bar{\bar{W}}_n$ for the correlated queue to that for the M/M/1 queue which are tabulated in Table 2b, are also plotted in Figure 4a and Figure 4b. In particular, Figure 4a shows that the correlation causes a decrease in mean waiting time for ρ small, and an increase when ρ is large. The latter effect probably occurs because the service times become highly autocorrelated as ρ increases.

The shape of the plots in Figure 4a suggested the possibility of fitting the mean waiting time to a function of

$\rho/(1-\rho)$ and $t/(1-t)$ so as to be able to predict E(W) for large values of t. The form of the independent variables were suggested by the fact that a log transform showed lnE(W) to go up approximately as $\ln(1-\rho)$ for large ρ, and the fact that for small ρ, E(W) is smaller than E(W) in the M/M/1 case. Also the form $t/(1-t)$ was suggested by the M/M/1 theory.

Only one value of β was used in the simulations, $\beta = 0.5$, so that the results did not reflect this variable. The rough fit obtained for t close to one was, for the MDxMD/1 queue with parameters $t, \rho, E(S)$.

$$\text{(V.1)} \qquad E^*(W) = \frac{t}{1-t}\left[0.5 + \frac{0.25\rho}{1-\rho}\right]E(S) .$$

One use for this formula would be to predict $E^*(W)$ for high t so that a simulation could be started with W_o exponential with mean $E(W_n)$ (the probability of a zero waiting time can be neglected for high t). This reduces the transient effect, as we will see later.

Formula (V.I) predicts that for $t = 0.99$, $\rho = 0.25$ and $E(S) = 0.25$, the mean waiting time would be

$$E^*(W) = 14.43 \ , \quad t = 0.99, \ \rho = 0.25, \ E(S) = 0.25$$

The mean for the M/M/1 queue is given as

$$E(W) = \frac{t}{1-t} E(S) = 24.75$$

A simulation was performed for this traffic intensity ($t = .99$), starting with $W_o = 0$; the simulation was found to converge to steady-state for $n = 280{,}000$ and then (see Figures 12a and 12b)

$$\bar{\bar{W}}_{280,000} = 14.194$$

with

$$\text{s.d.}(\bar{\bar{W}}_{280,000}) = 0.1731.$$

This is in reasonable agreement.

The effect of using this value for $E(W_o)$ in the simulation is discussed in Section V.D.

Traffic Intensity, t

Correlation, ρ	.25	.50	.75	.90
.00	0.073763 (0.000064)	0.192998 (0.000156)	0.484147 (0.000501)	1.258926 (0.003004)
.25	0.077745 (0.000730)	0.210338 (0.000165)	0.543177 (0.000597)	1.444641 (0.003373)
.50	0.083026 (0.000091)	0.237792 (0.000230)	0.649296 (0.000899)	1.798944 (0.004716)
.75	0.091539 (0.000130)	0.296498 (0.000440)	0.926665 (0.001709)	2.819429 (0.009518)
.90	0.101673 (0.000188)	0.413239 (0.000950)	1.654127 (0.004754)	5.717662 (0.032587)
M/M/1 Queue	0.083365 (0.000076)	0.250211 (0.000231)	0.749701 (0.000983)	2.254441 (0.007270)

Table 2a. Estimated mean $\bar{\bar{W}}_n$ and standard deviation of the estimated mean of the average, steady state waiting time in the correlated queue for various ρ and t ($\beta = 0.5$ and $\mu_S = 0.25$). The estimated s.d. of $\bar{\bar{W}}_n$ is given in brackets.

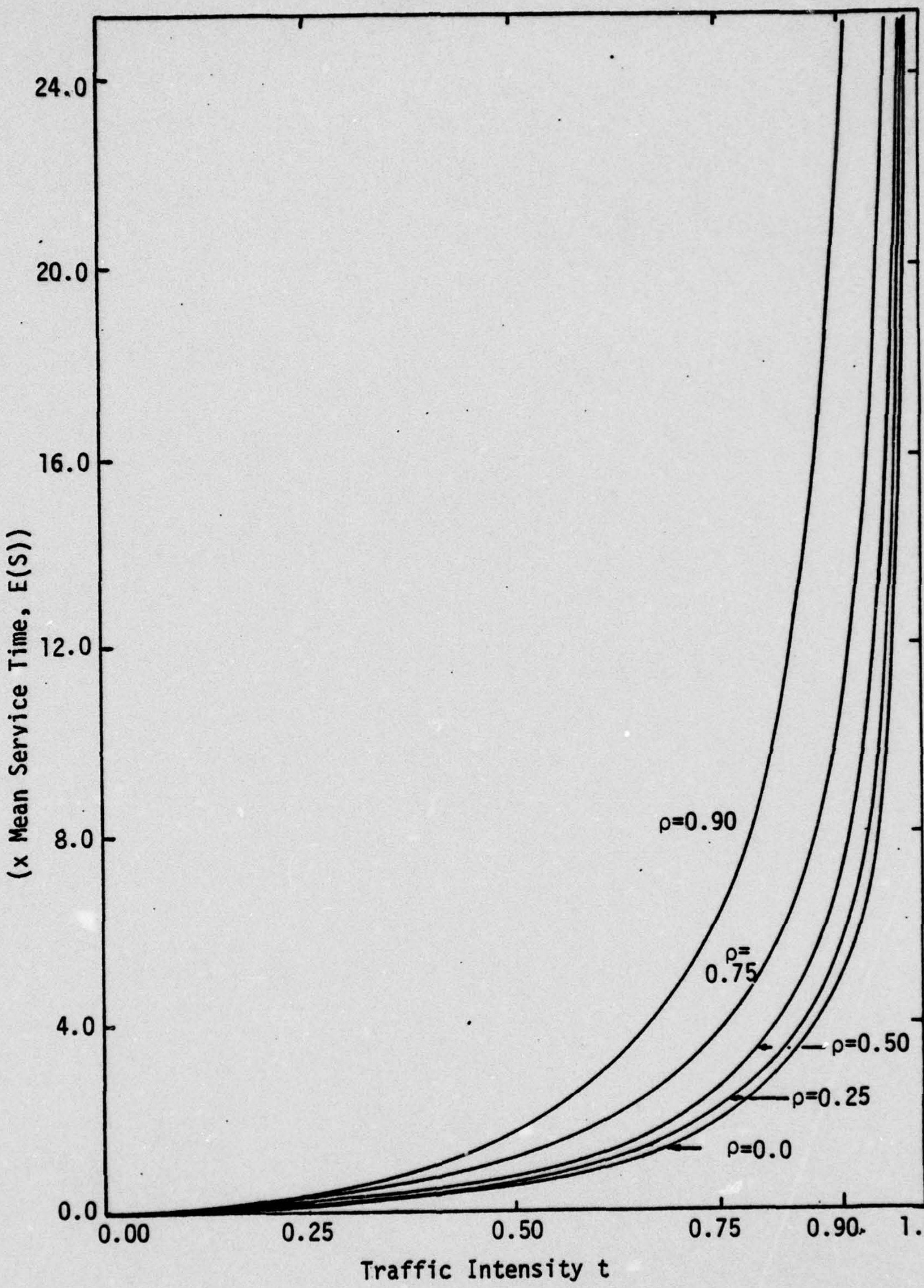


Figure 3a. Smoothed, esimated mean waiting time in the correlated MD*MD/1 queue as a function of traffic intensity, t. In units of mean service time for several values of the correlation paramter ρ. Fixed β = 0.5.

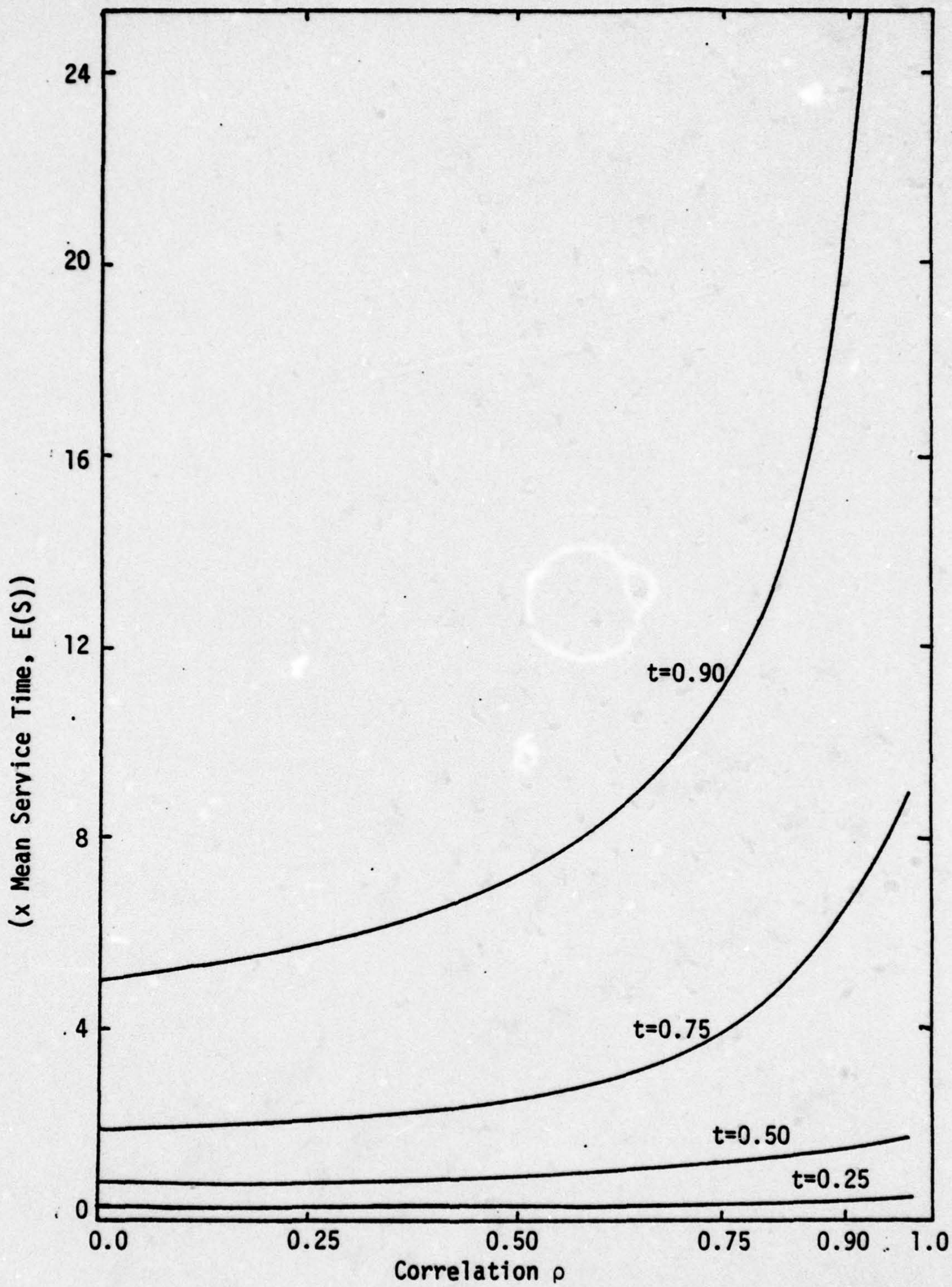


Figure 3b. Smoothed, estimated mean waiting time in the correlated MDxMD/1 queue as a function of the correlation parameter ρ. In units of mean service time for several values of the traffic intensity t. Fixed β = 0.5.

Traffic Intensity, t

Correlation, ρ	.25	.50	.75	.90
.00	0.8852	0.7720	0.6455	0.5595
.25	0.9329	0.8414	0.7242	0.6421
.50	0.9963	0.9512	0.8657	0.7995
.75	1.0985	1.1860	1.2356	1.2531
.90	1.2201	1.6530	2.2055	2.5412

Table 2b. Ratio of the Estimated mean $\bar{\bar{W}}_n$ of the average waiting time in the correlated queue to the known average waiting time of the uncorrelated (M/M/1) queue ($\beta = 0.5$) for various values of ρ and t.

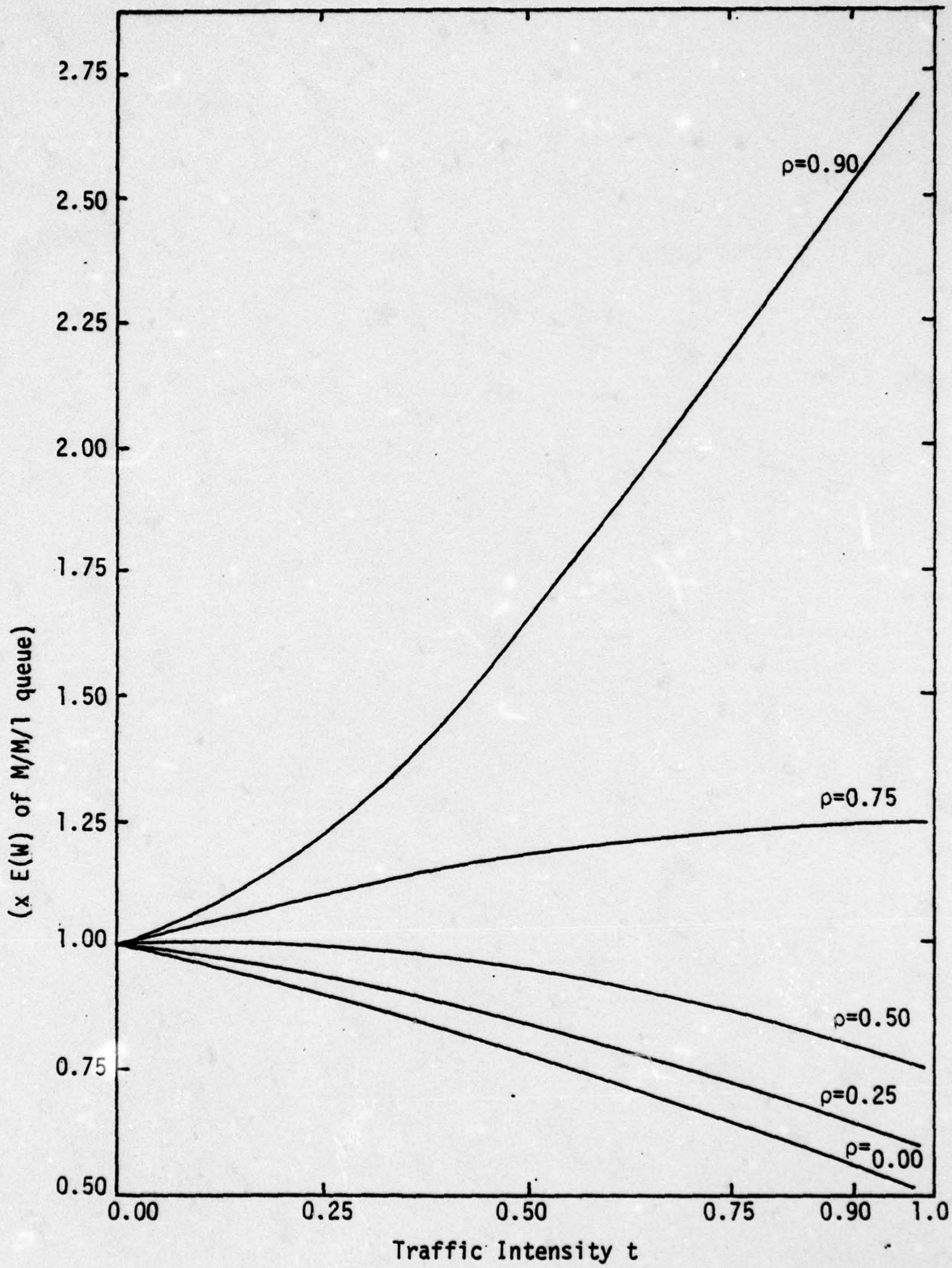


Figure 4a. Smoothed, estimated mean waiting time in the correlated MDxMD/1 queue divided by the known mean waiting time in the M/M/1 queue <u>as a function of traffic intensity t</u>. Thus the result is in units of the mean waiting time in M/M/1 queue. Here the correlation parameter ρ is the parameter. Fixed $\beta = 0.5$

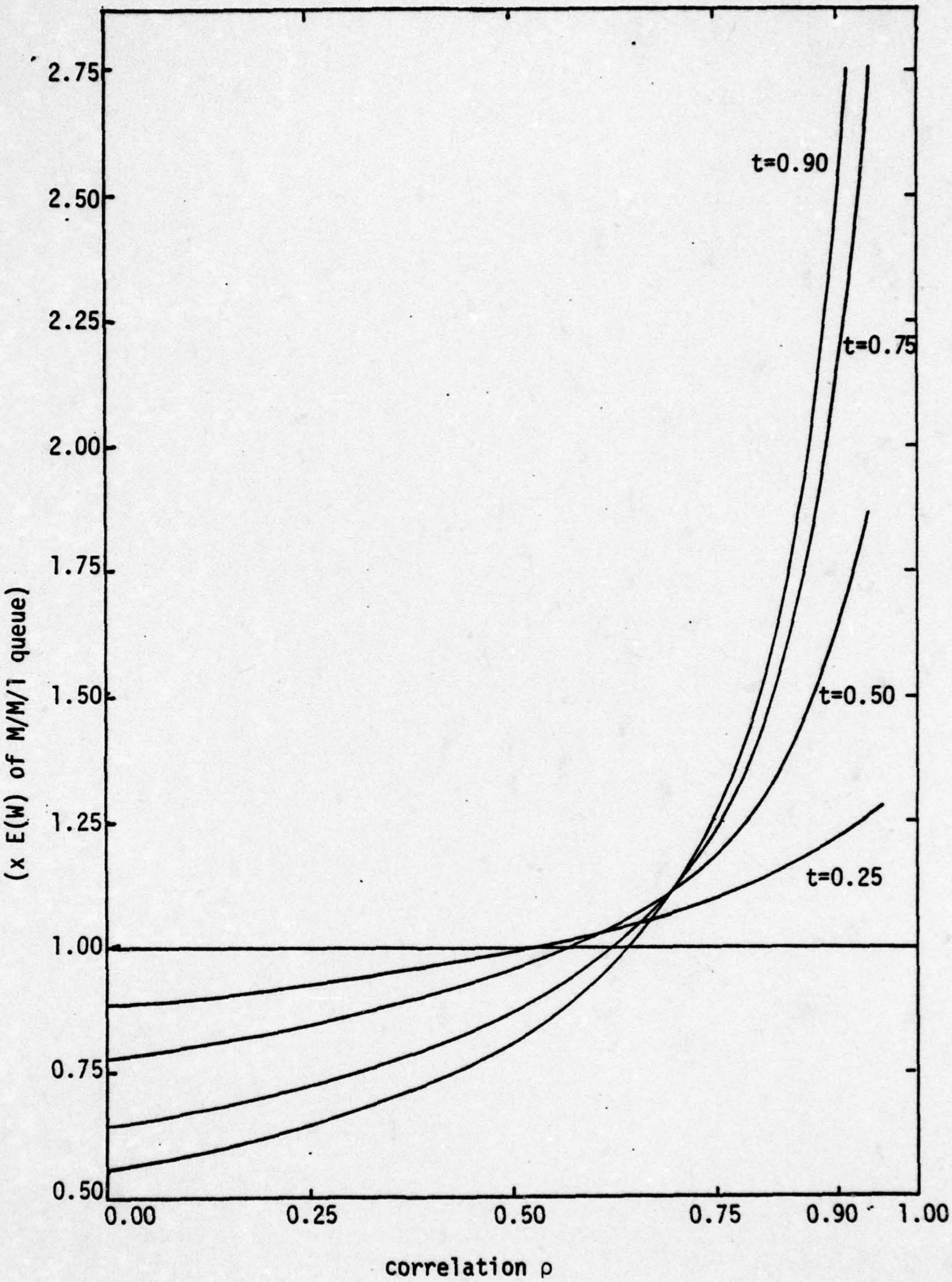


Figure 4b. Smoothed, estimated mean waiting time in the correlated $MD^xMD/1$ queue divided by the known mean waiting time in the M/M/1 queue as a function of the correlation parameter ρ. The result is in units of the mean waiting time in M/M/1 queue. Here the traffic intensity t is the parameter. Fixed $\beta = 0.5$.

D. ESTIMATED DISTRIBUTION OF WAITING TIME W

The simulation program was set up to enable us to look at statistics on W_n and $\bar{W}_n$ across the replications at various n's. In addition the file system enabled us to look at joing properties of the samples at two different n's, and in particular at correlations. The reason for this is that a sample of size 500 is rather small when one wants to look at the steady state distribution of W_n, namely W. One could increase the number of replications, but this is constrained by the size of the computer. However, since the simulation has to be carried out well beyond the set in of the steady state for safety sake, one might as well pool samples of waiting times along the sample path if they are sufficiently far apart to be uncorrelated, or approximately so. Then instead of looking at a sample size of 500, we may pool waiting times at 10 steps of n, n_1, n_2, ...,n_{10}, to make a sample size of 5000. Then the waiting time distribution for a particular run can be shown as in fig. 5a, 5b; these are actually histograms showing many of the estimated sample statistics which might be of interest.

In each step of the simulation runs which were performed the correlation between waiting times 1000 arrivals apart were obtained and found to be negligible, (see Table 3a, 3b),

n = no. of arrivals	6000	7000	8000	9000	10000
6000	1.0	-0.0118	0.0135	0.0002	-0.0384
7000	-0.0118	1.0	-0.0450	0.0403	-0.1079
8000	0.0135	-0.0450	1.0	0.0012	-0.0088
9000	0.0002	0.0403	0.0012	1.0	-0.0491
10000	-0.0384	-0.1079	-0.0088	-0.0491	1.0

Table 3a. Correlation Matrix of Waiting Times for Different Indices n, in the Correlated Queue. $t = 0.75$, $\beta = 0.50$, $\rho = 0.75$.

n = no. of arrivals	6000	7000	8000	9000	10000
6000	1.0	-0.0371	0.0280	0.0105	-0.0131
7000	-0.0371	1.0	-0.0413	0.0779	-0.1707
8000	0.0280	-0.0413	1.0	0.0189	-0.0241
9000	0.0105	0.0779	0.0189	1.0	-0.0977
10000	-0.0131	-0.1707	-0.0241	-0.0977	1.0

Table 3b. Correlation Matrix of Positive Waiting Times (No Zeros) for Different Indices n in the Correlated Queue. $t = .75$, $\beta = .50$, $\rho = 0.75$.

thus justifying the pooled samples. Of course a more efficient use of the data would have been to calculate a correlogram of estimated serial correlations along each (stationary) sample path and average over the five hundred replications to obtain a more precise estimate of each serial correlation together with a variance estimate of that estimate.

Another reason for looking at the samples $W_n(j)$, $j = 1, \ldots, 500$ at different values of n is to assess the convergence to a steady state in terms of the whole distribution of W_n, rather than just its mean value. There are a number of ways of doing this display and a particularly simple and graphic way is to give box-plots for the successive samples on the same scale. Unfortunately, this proved to be difficult with presently available software;; thus in Figs. 6a-g we give box plots of $W_n(j)$ for n = 1,000(1,000)70,000. The scales in these figures are unfortunately not commensurate, but the convergence to steady state can be seen. Note that unlike the $\overline{W}_n(j)$'s, the variance of the $W_n(j)$'s are not decreasing with n; in fact that should converge to var(W). If some of the distributional detail in the plots is overwhelming, one can look solely at the *'s in the boxes, which are the median values in the successive samples. Since these samples are, as we have seen, approximately uncorrelated, the *'s could have been smoothed.

The plots in Figures 6a-g are tied to zero by the occurrence of zero waiting times (on the average (1-t)% of the sample). Thus plots of the positive waiting times are given in Figures 7a-g.

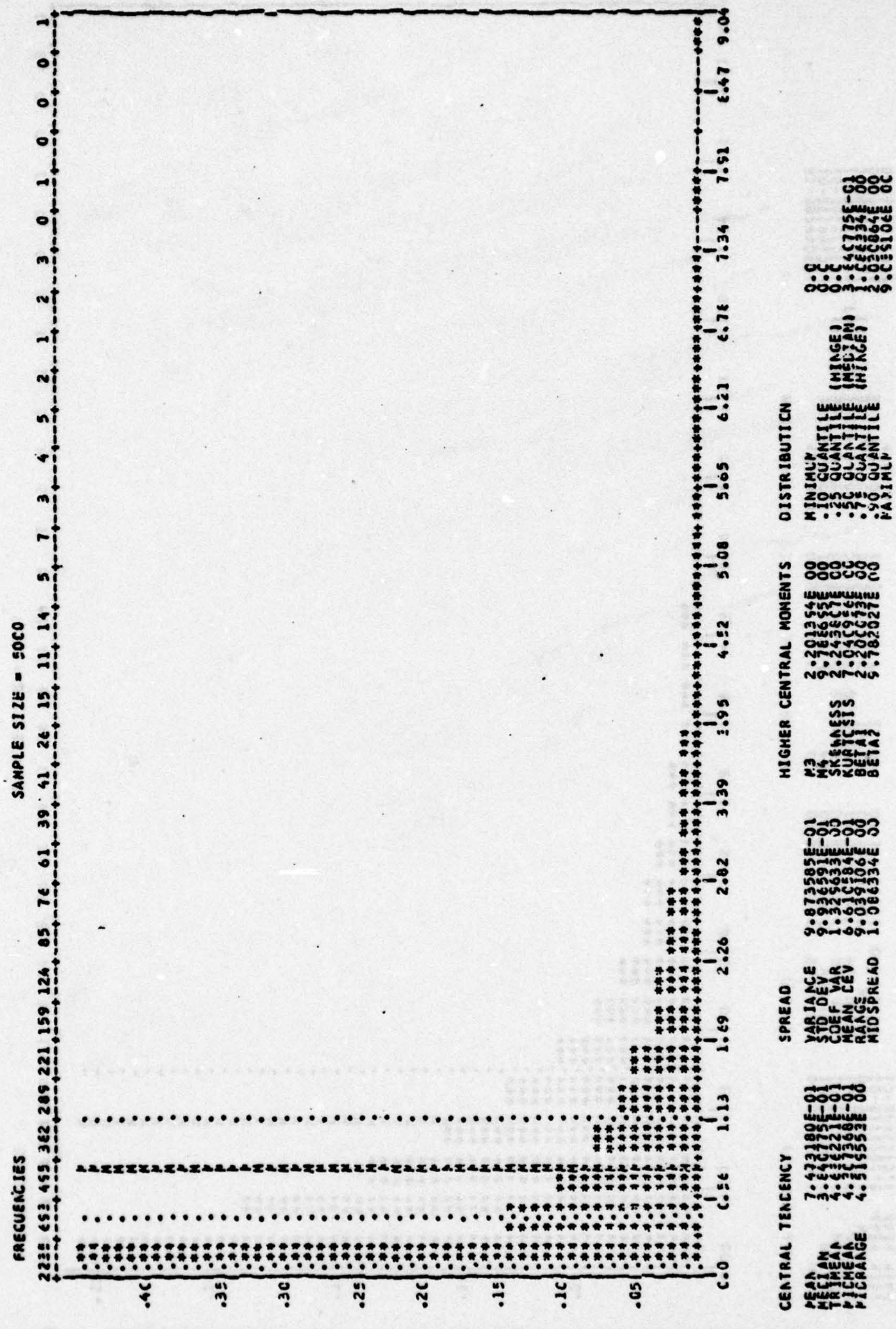


Figure 5a. Histogram and sample statistics for combined waiting time data in the simulated, uncorrelated M/M/1 queue, i.e. all $W_n(j)$'s for j=1,...,500 and n = 51,000(1000)60,000. t = 0.75, 1/E(S) = 4.0

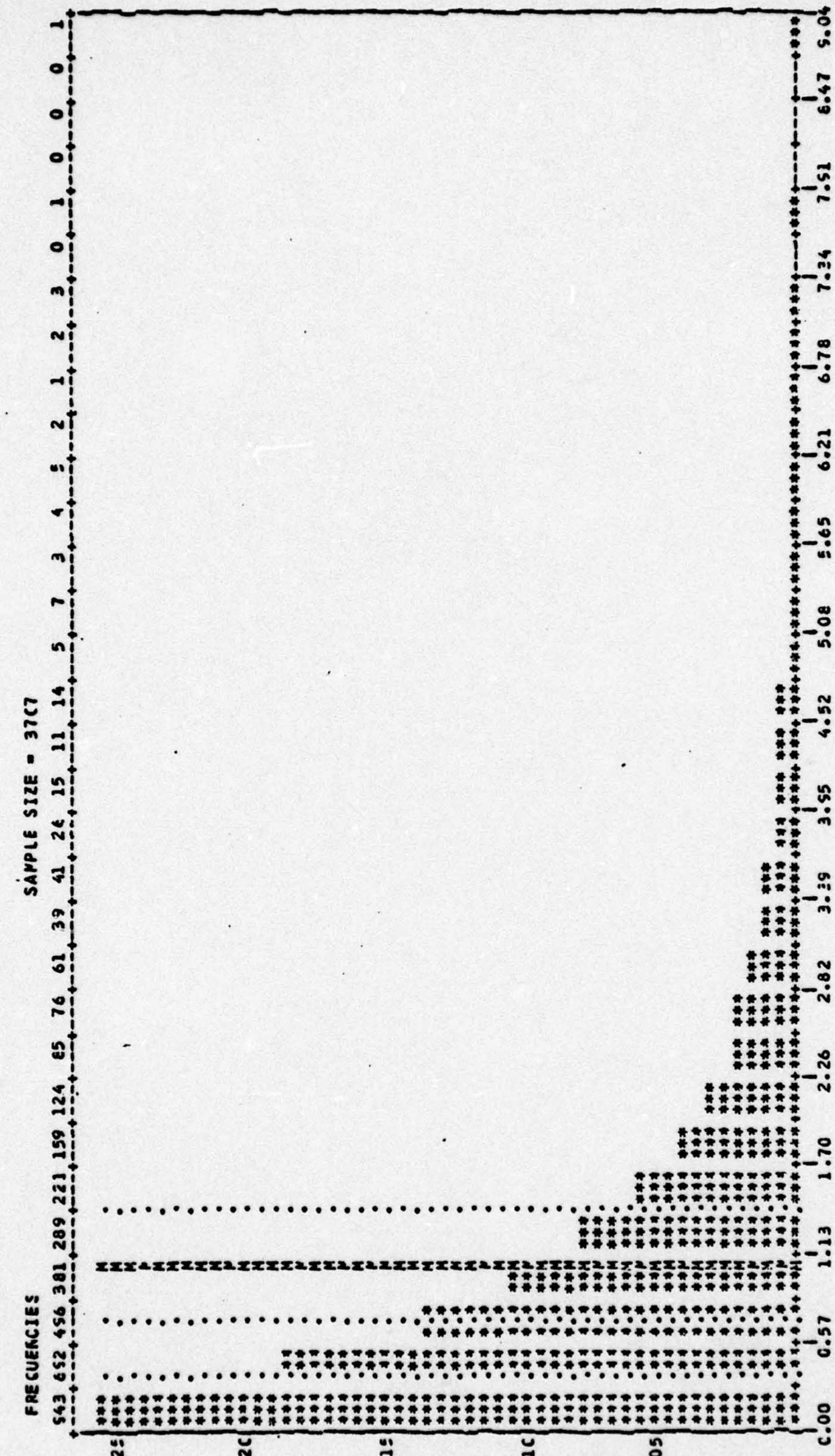


Figure 5b. Histogram and sample statistics for combined waiting time data (with 2,293 zero waiting times removed) in the simulated, uncorrelated M/M/1 queue, i.e. all $W_n(j)$'s for $j=1,\ldots,500$ and $n = 51{,}000(1000)60{,}000$. $t = 0.75$, $1/E(S) = 4.0$

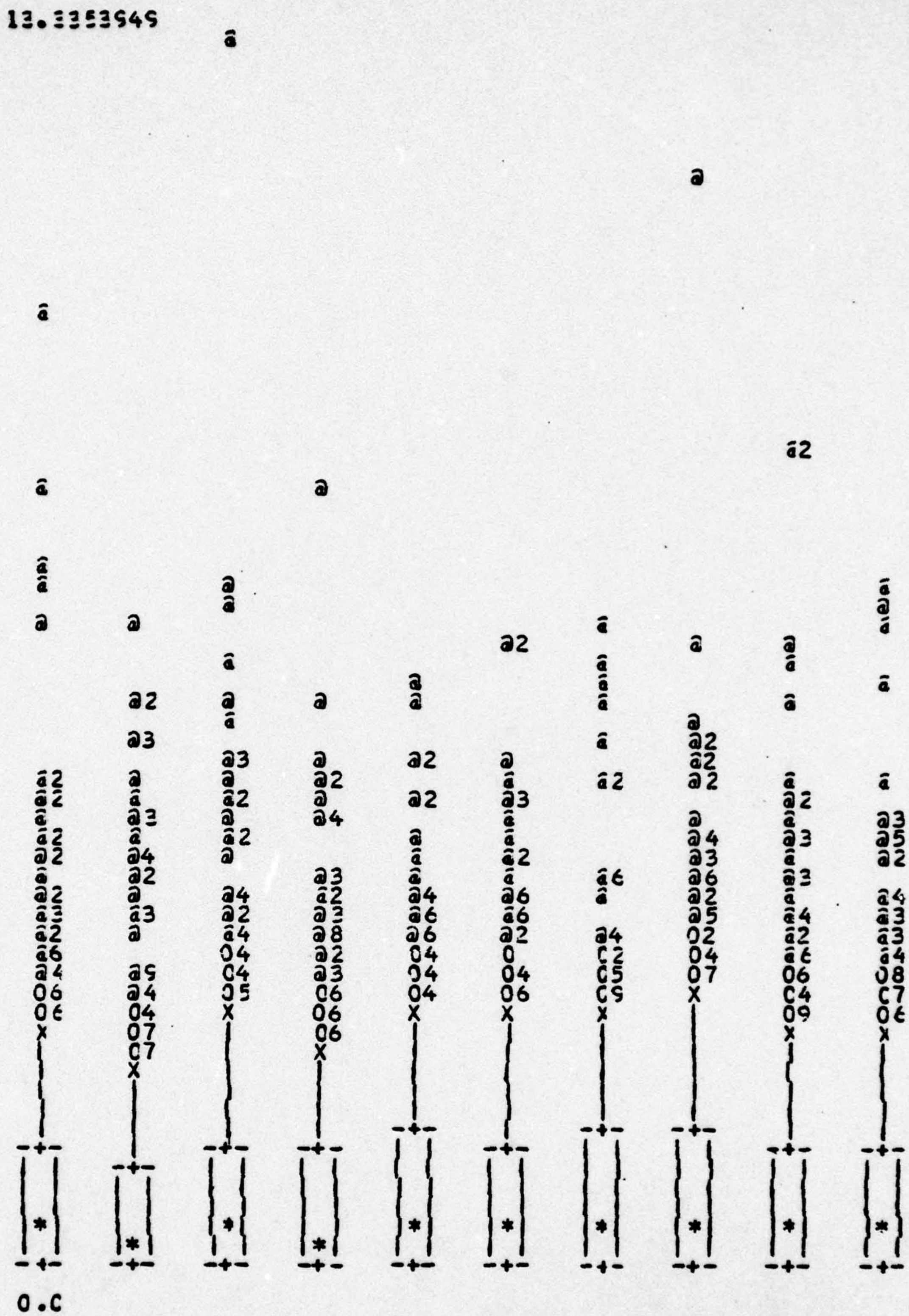


Figure 6a. Box plots of the sample waiting times $W_n(j)$, $j=1,\ldots,500$ for values of n = 1,000 to 10,000 in steps of 1,000. Correlated queue; $t = 0.75$, $\beta = 0.5$, $\rho = 0.75$. Figures at top and bottom at left are maximum and minimum sample values.

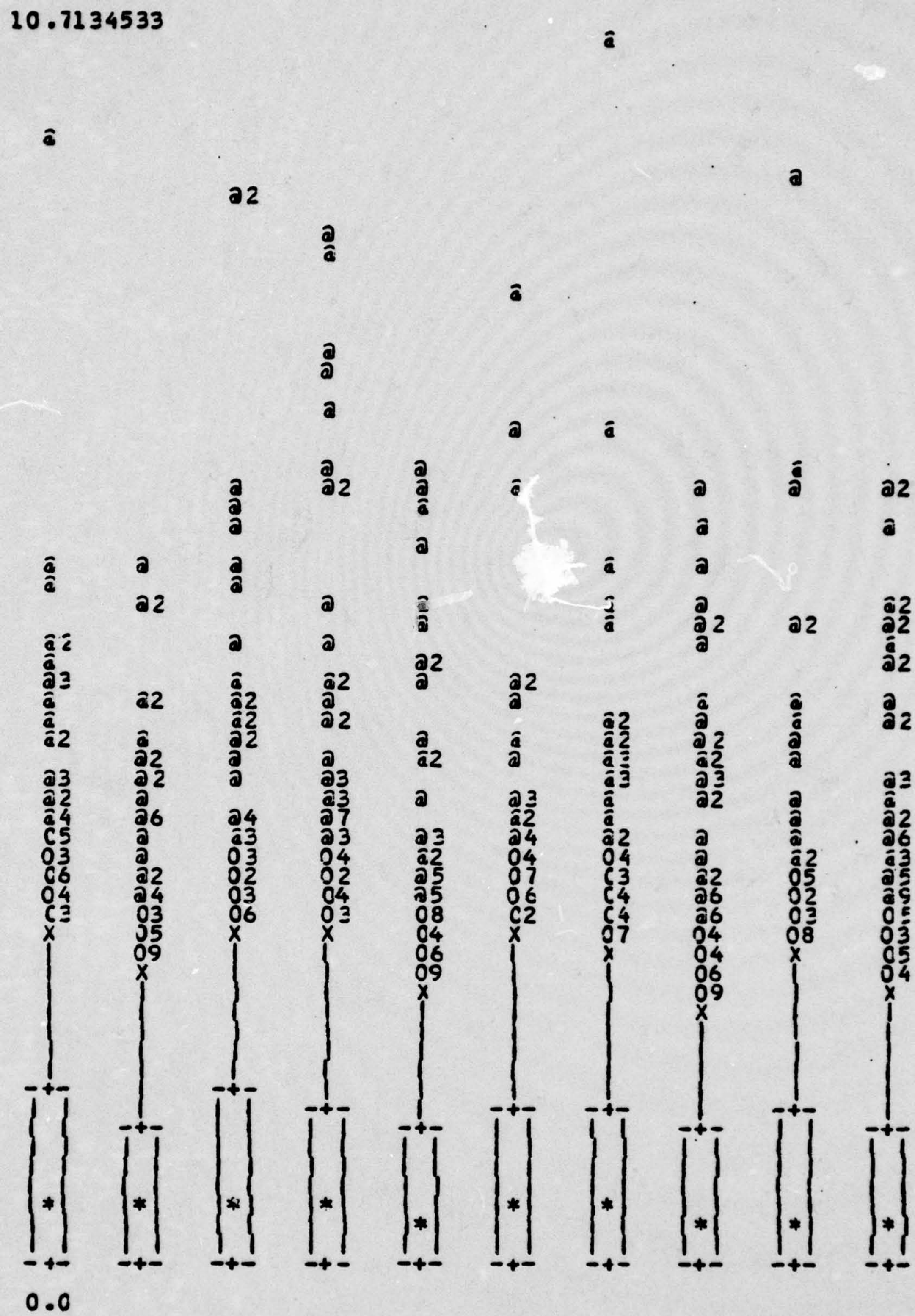


Figure 6b. Box plots of the sample waiting times $W_n(j)$, j=1,...,500 for values of n = 11,000 to 20,000 in steps of 1,000. Correlated queue; t = 0.75, β = 0.5, ρ = 0.75. Figures at top and bottom at left are maximum and minimum sample values.

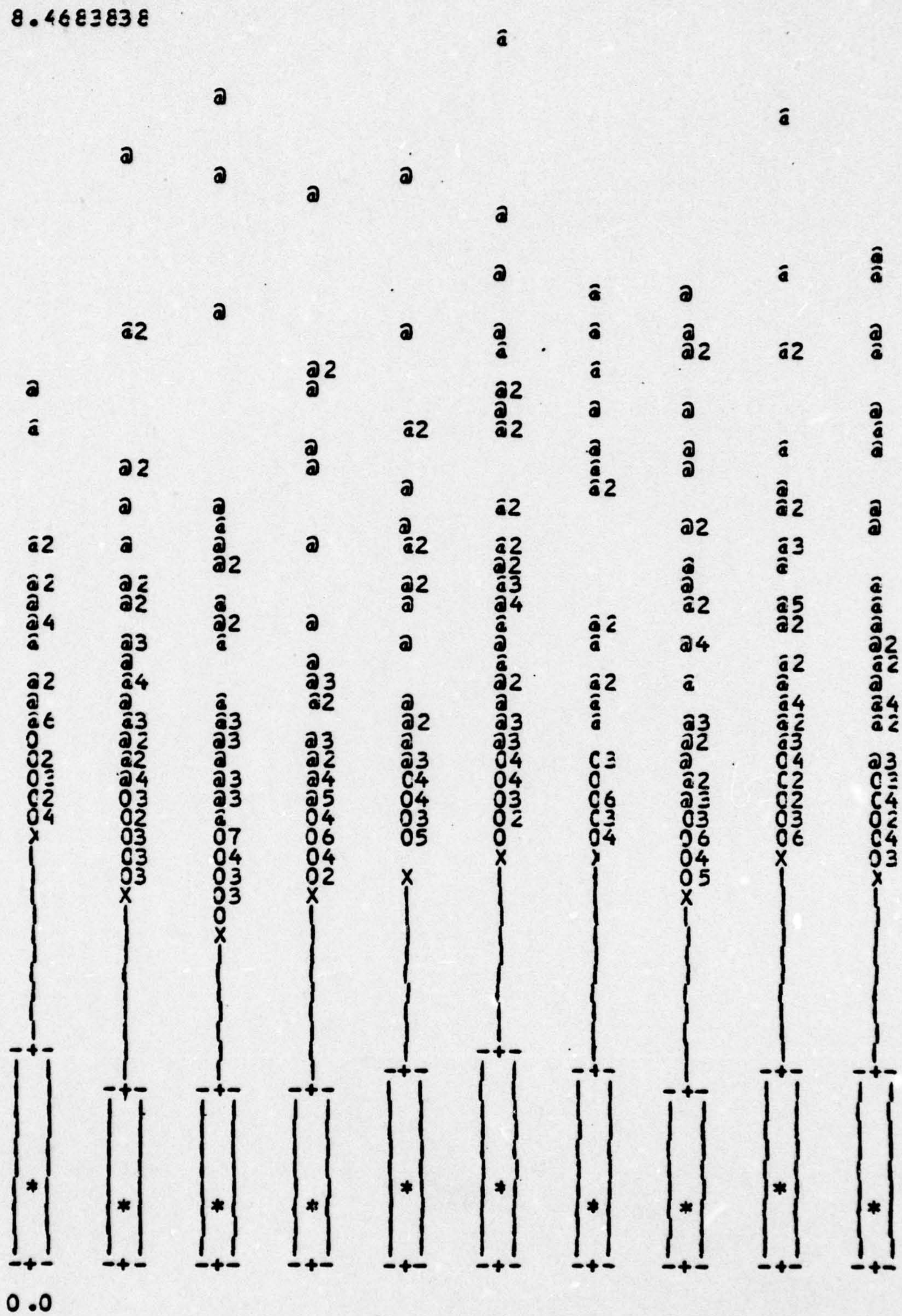


Figure 6c. Box plots of the sample waiting times $W_n(j)$, j=1,...,500 for values of n = 21,000 to 30,000 in steps of 1,000. Correlated queue; t = 0.75, β = 0.5, ρ = 0.75. Figures at top and bottom at left are maximum and minimum sample values.

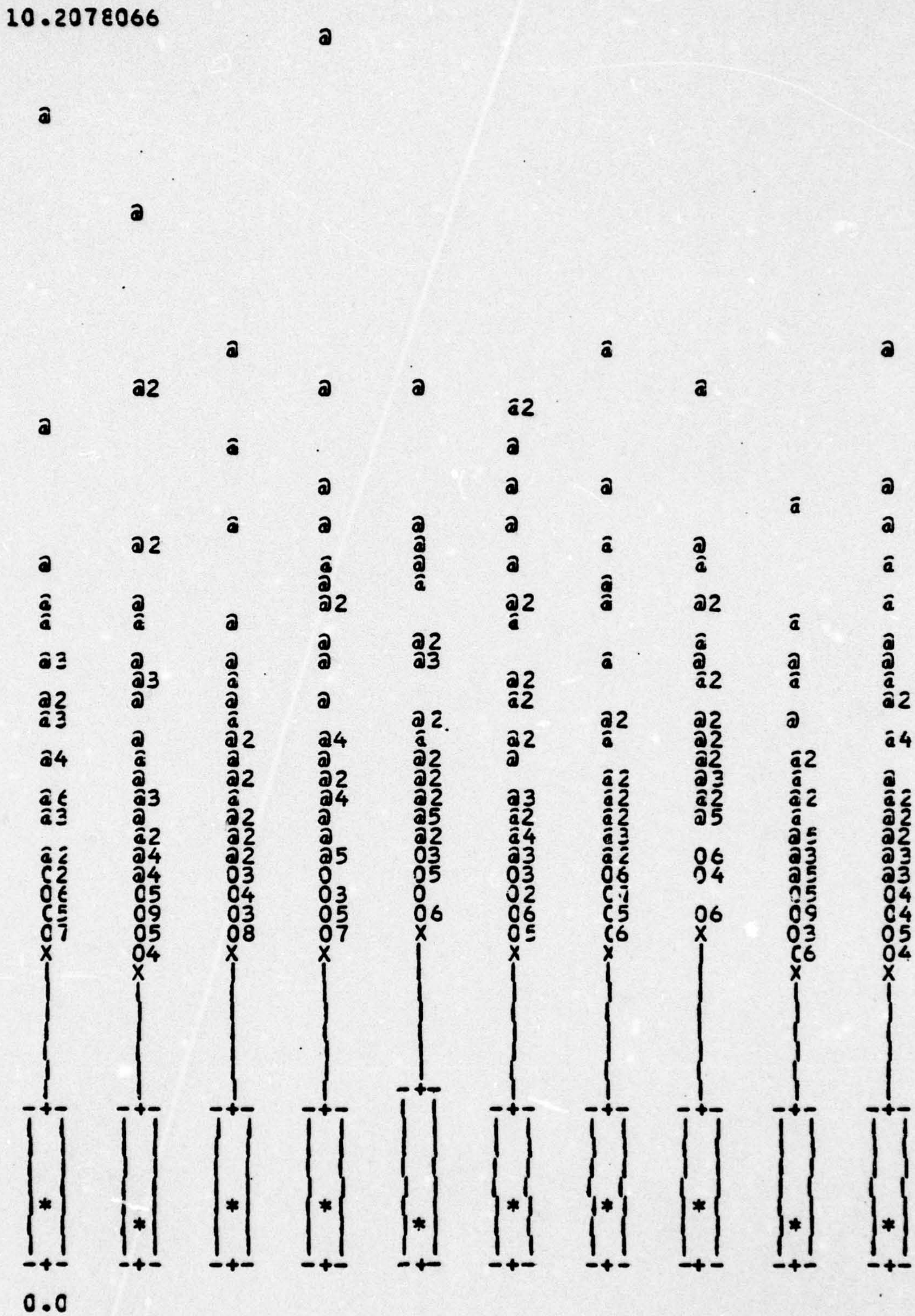


Figure 6d. Box plots of sample waiting times $W_n(j)$, j=1,...,500 for values of n = 31,000 to 40,000 in steps of 1,000. <u>Correlated</u> <u>queue</u>; t = 0.75, β = 0.5, ρ = 0.75. Figures at top and bottom at left are maximum and minimum sample values.

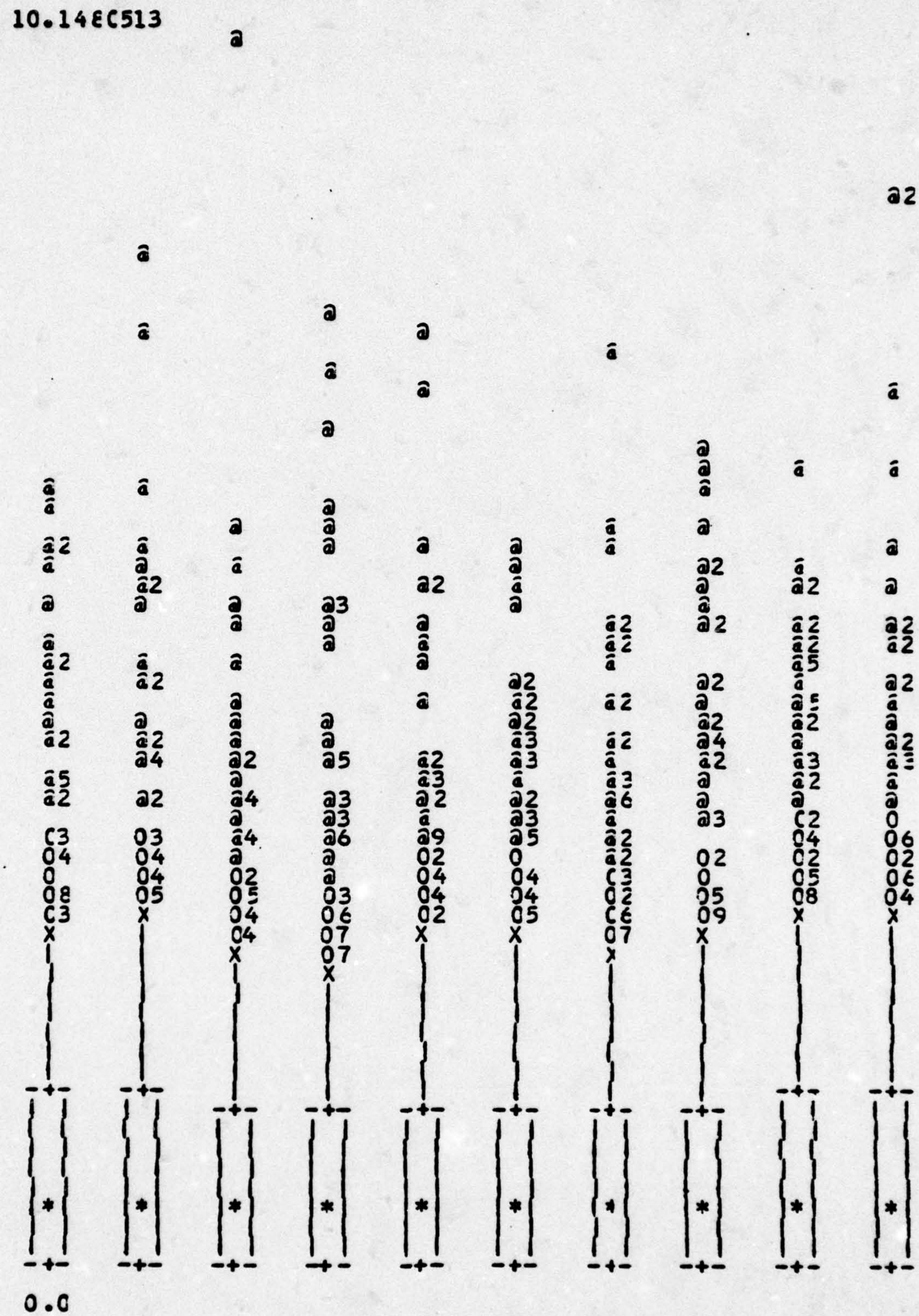


Figure 6e. Box plots of sample waiting times $W_n(j)$, $j=1,\ldots,500$ for values of n = 41,000 to 50,000 in steps of 1,000. Correlated queue; $t = 0.75$, $\beta = 0.5$, $\rho = 0.75$. Figures at top and bottom at left are maximum and minimum sample values.

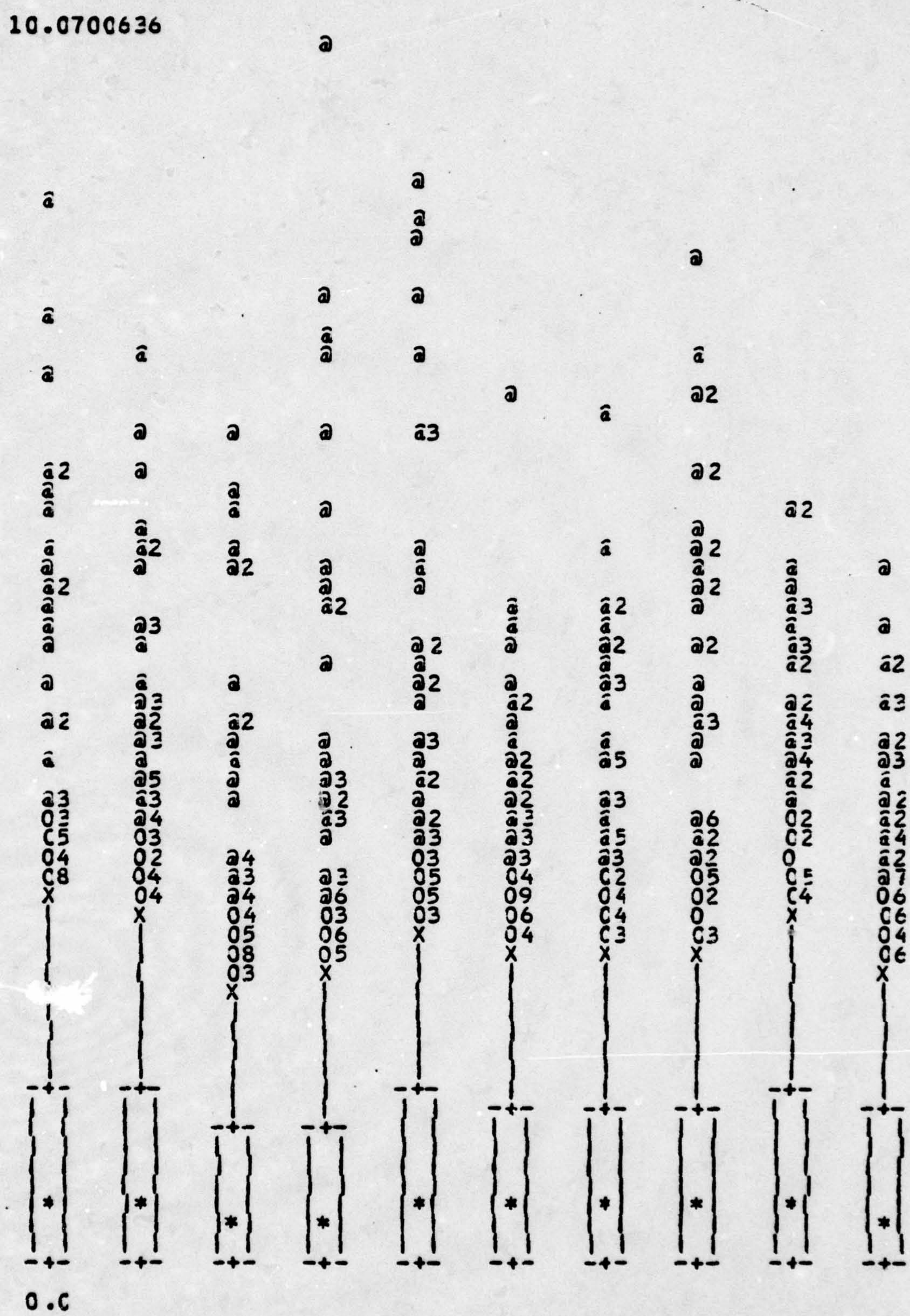


Figure 6f. Box plots of sample waiting times $W_n(j)$, j=1,...,500 for values of n = 51,000 to 60,000 in steps of 1,000. Correlated queue; t = 0.75, β = 0.5, ρ = 0.75. Figures at top and bottom at left are maximum and minimum sample values.

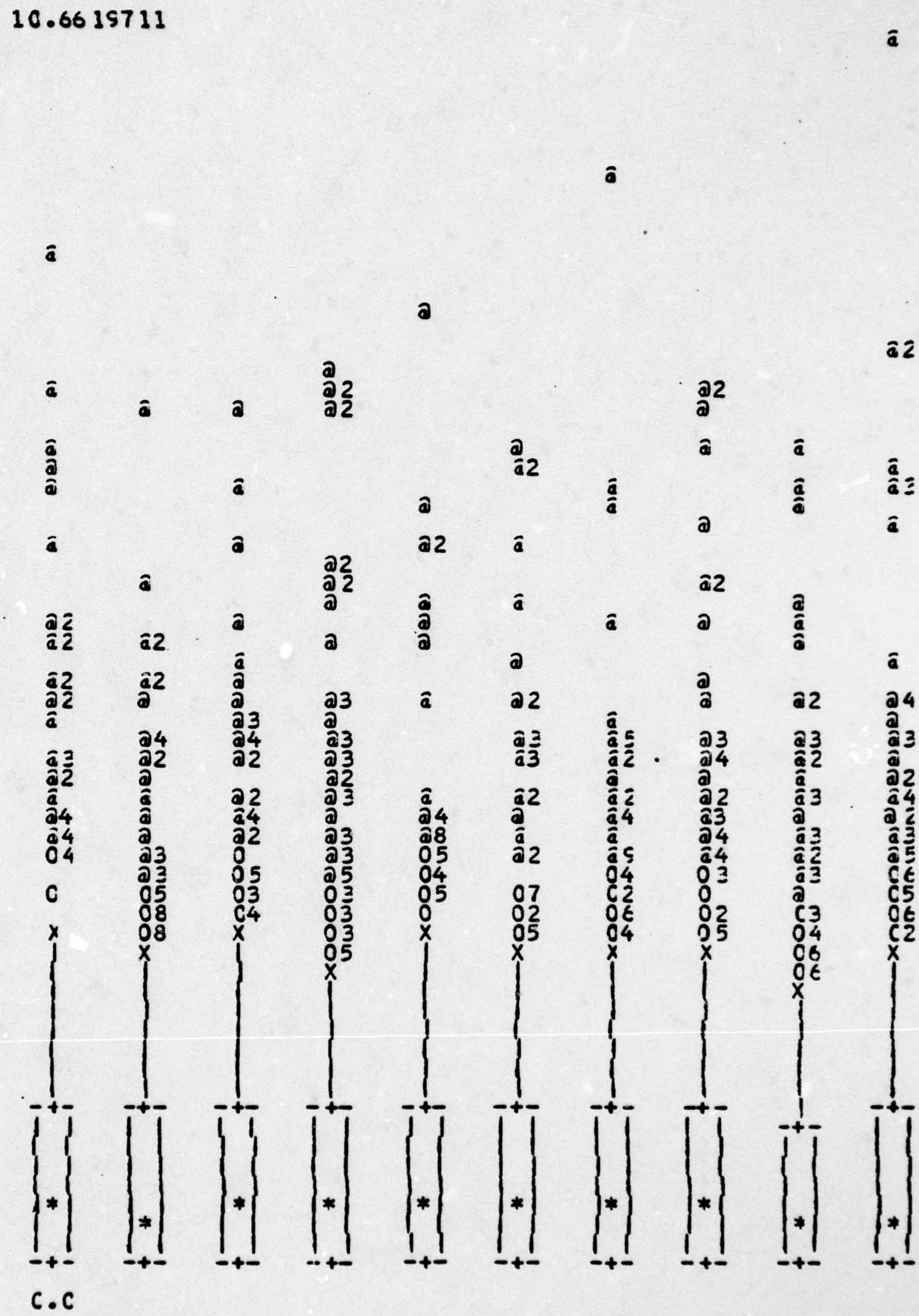


Figure 6g. Box plots of sample waiting times $W_n(j)$, j=1,...,500 for values of n = 61,000 to 70,000 in steps of 1,000. <u>Correlated</u> <u>queue</u>; $t = 0.75$, $\beta = 0.5$, $\rho = 0.75$. Figures at top and bottom at left are maximum and minimum sample values.

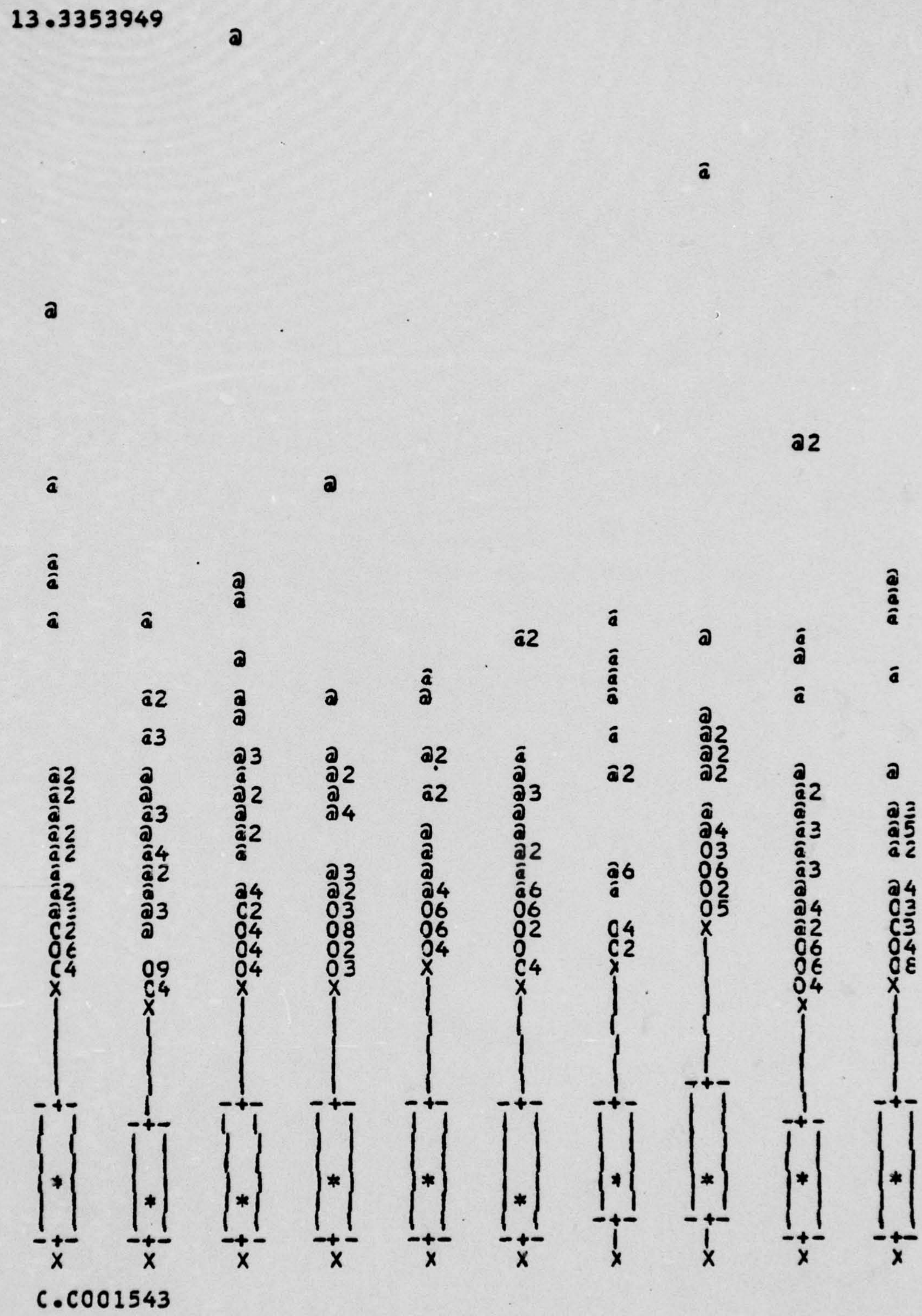


Figure 7a. Box plots of the sample waiting times $W_n(j)$, j=1,...,500 <u>with zero waiting times removed</u> for values of n = 1,000 to 10,000 in steps of 1,000. <u>Correlated queue</u>; $t = 0.75$, $\beta = 0.5$, $\rho = 0.75$. Figures at top and bottom at left are maximum and minimum sample values.

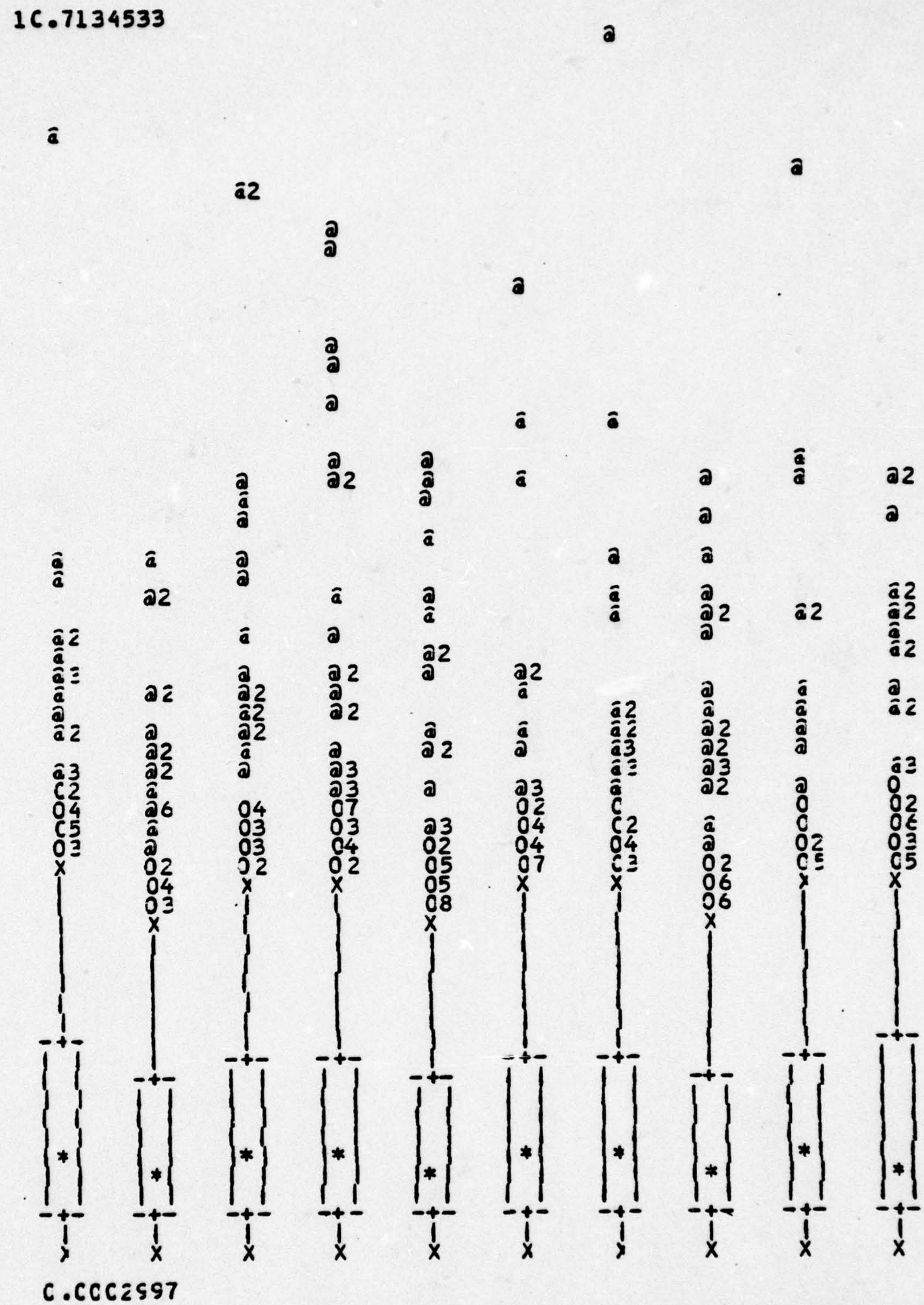


Figure 7b. Box plots of the sample waiting times $W_n(j)$, $j=1,\ldots,500$ with zero waiting times removed for values of $n = 11{,}000$ to $20{,}000$ in steps of 1,000. Correlated queue; $t = 0.75$, $\beta = 0.5$, $\rho = 0.75$. Figures at top and bottom at left are maximum and minimum sample values.

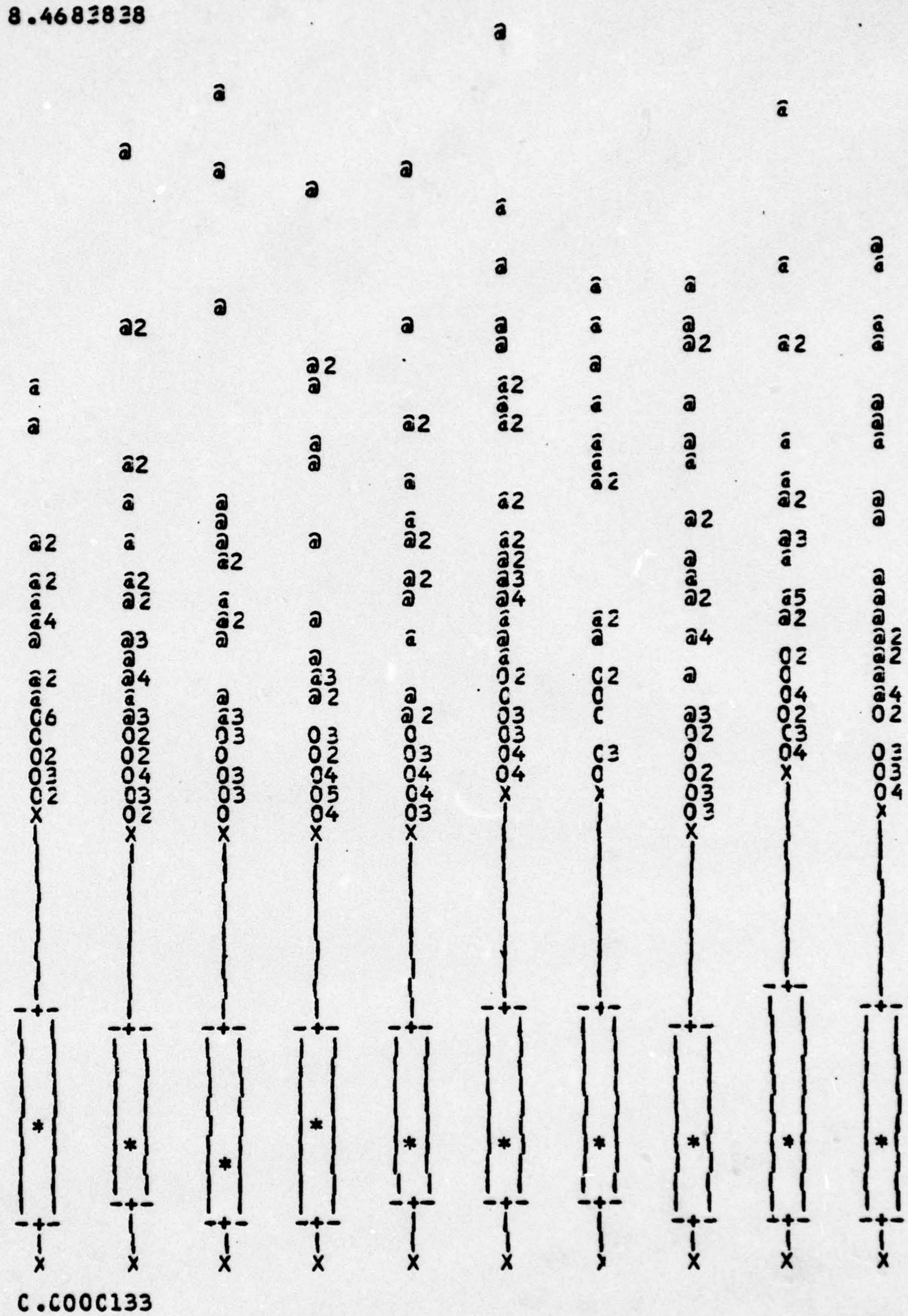


Figure 7c. Box plots of the sample waiting times $W_n(j)$, $j=1,\ldots,500$ with zero waiting times removed for values of n = 21,000 to 30,000 in steps of 1,000. Correlated queue; $t = 0.75$, $\beta = 0.5$, $\rho = 0.75$. Figures at top and bottom at left are maximum and minimum sample values.

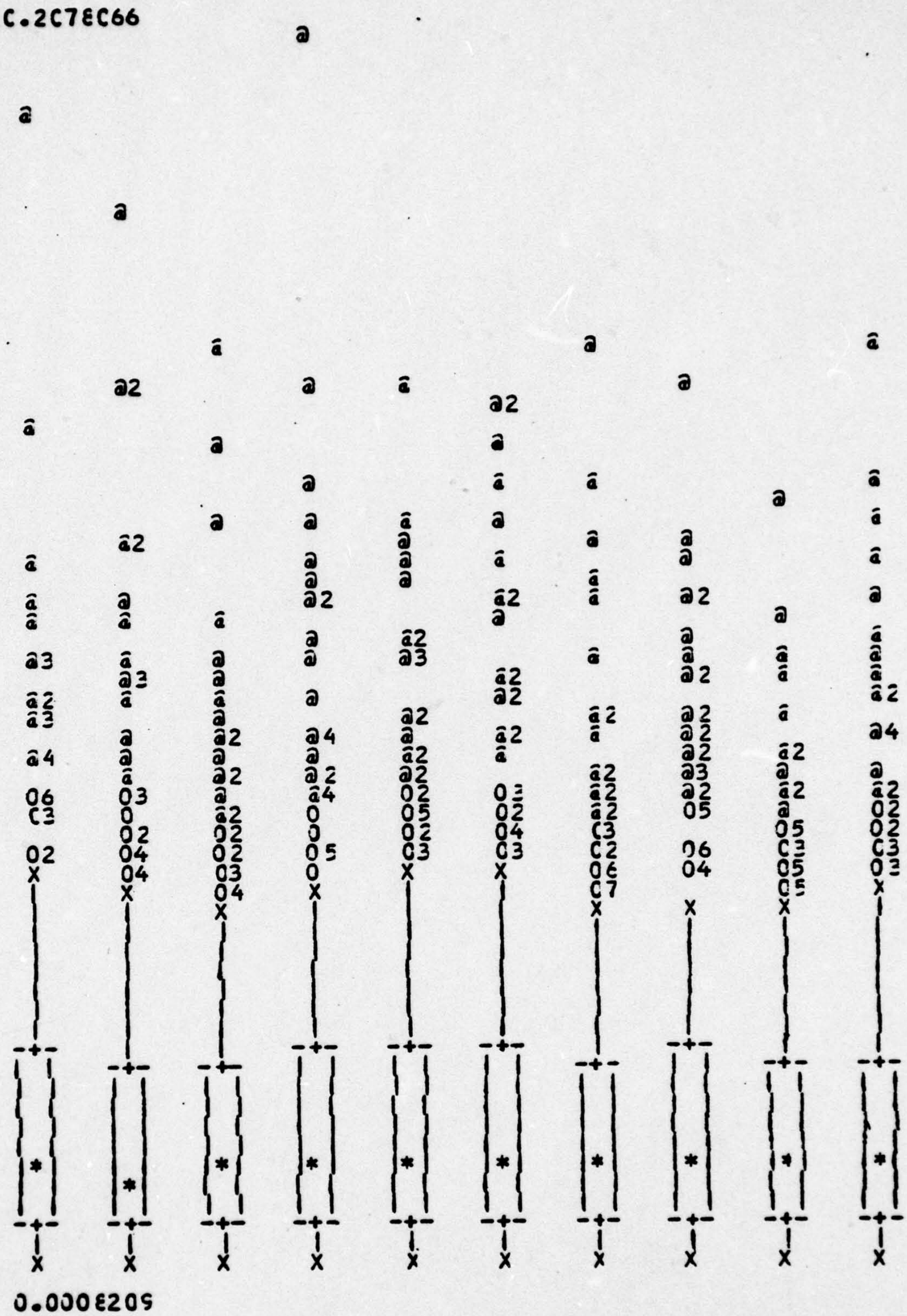


Figure 7d. Box plots of the sample waiting times $W_n(j)$, j=1,...,500 with zero waiting times removed for values of n = 31,000 to 40,000 in steps of 1,000. Correlated queue; t = 0.75, β = 0.5, ρ = 0.75. Figures at top and bottom at left are maximum and minimum sample values.

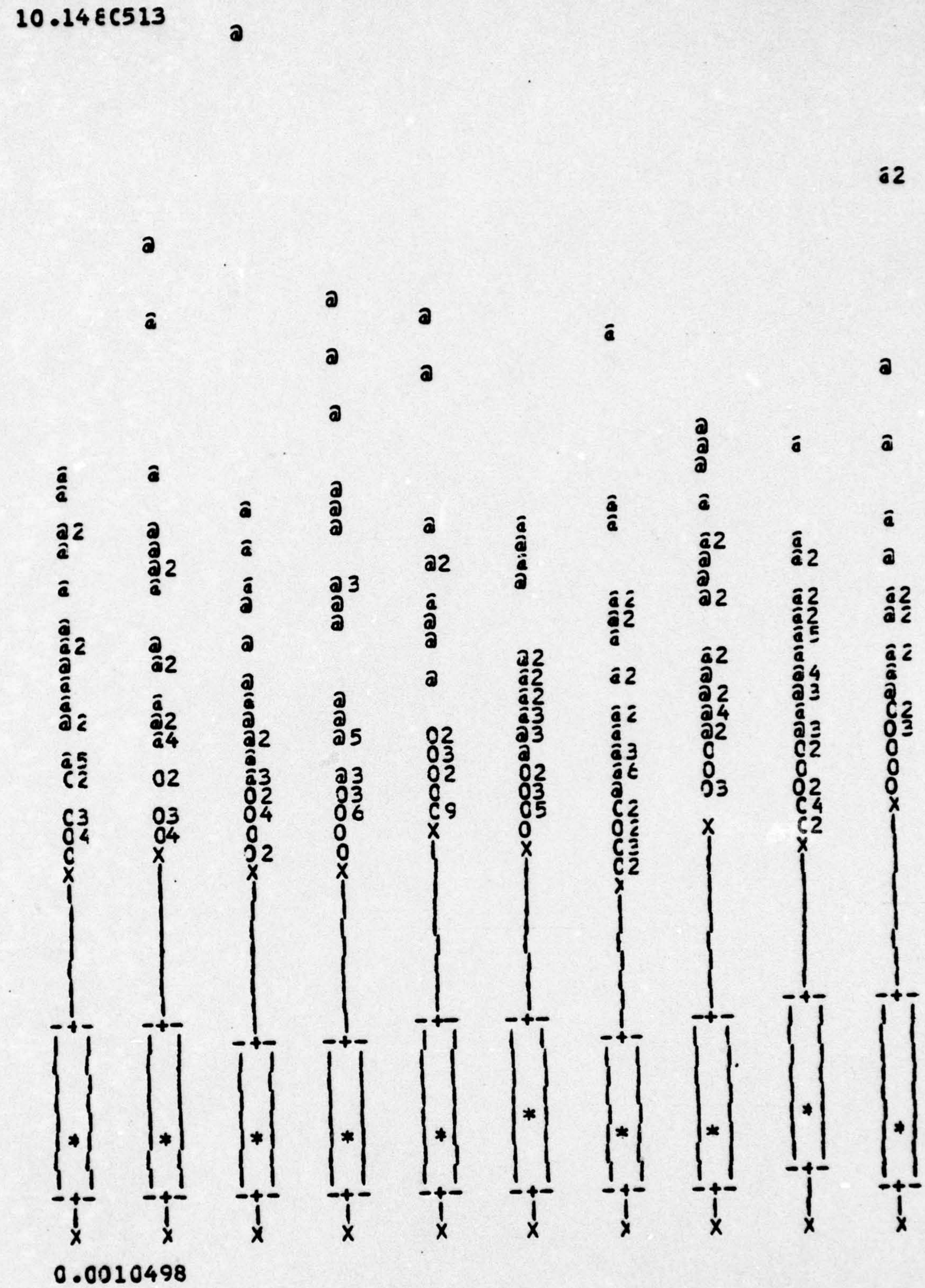


Figure 7e. Box plots of the sample waiting times $W_n(j)$, j=1,...,500 <u>with zero waiting times removed</u> for values of n = 41,000 to 50,000 in steps of 1,000. <u>Correlated queue</u>; t ≐ 0.75, β = 0.5, ρ = 0.75. Figures at top and bottom at left are maximum and minimum sample values.

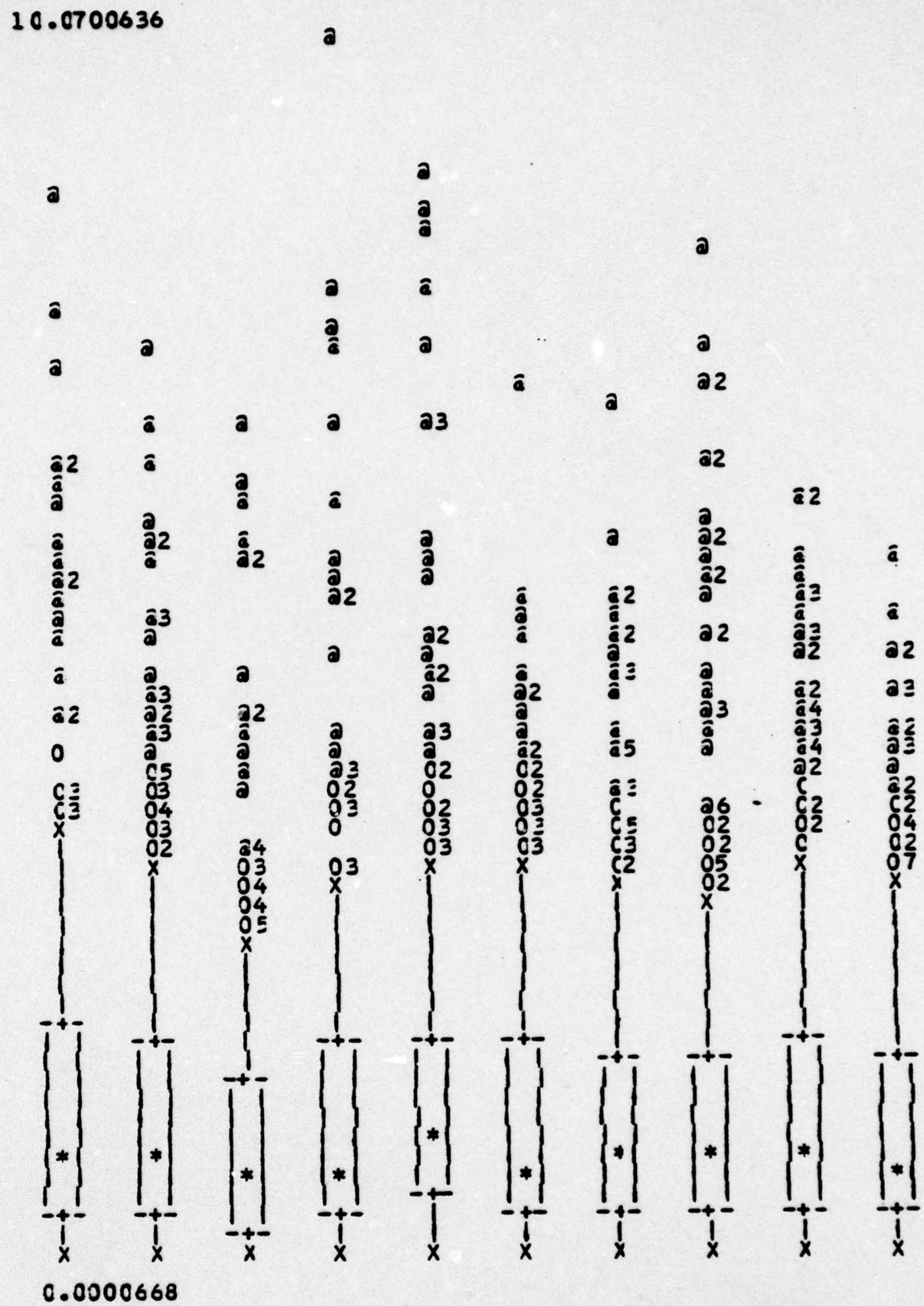


Figure 7f. Box plots of the sample waiting times $W_n(j)$, $j=1,\ldots,500$ with zero waiting times removed for values of $n = 51{,}000$ to $60{,}000$ in steps of 1,000. Correlated queue; $t = 0.75$, $\beta = 0.5$, $\rho = 0.75$. Figures at top and bottom at left are maximum and minimum sample values.

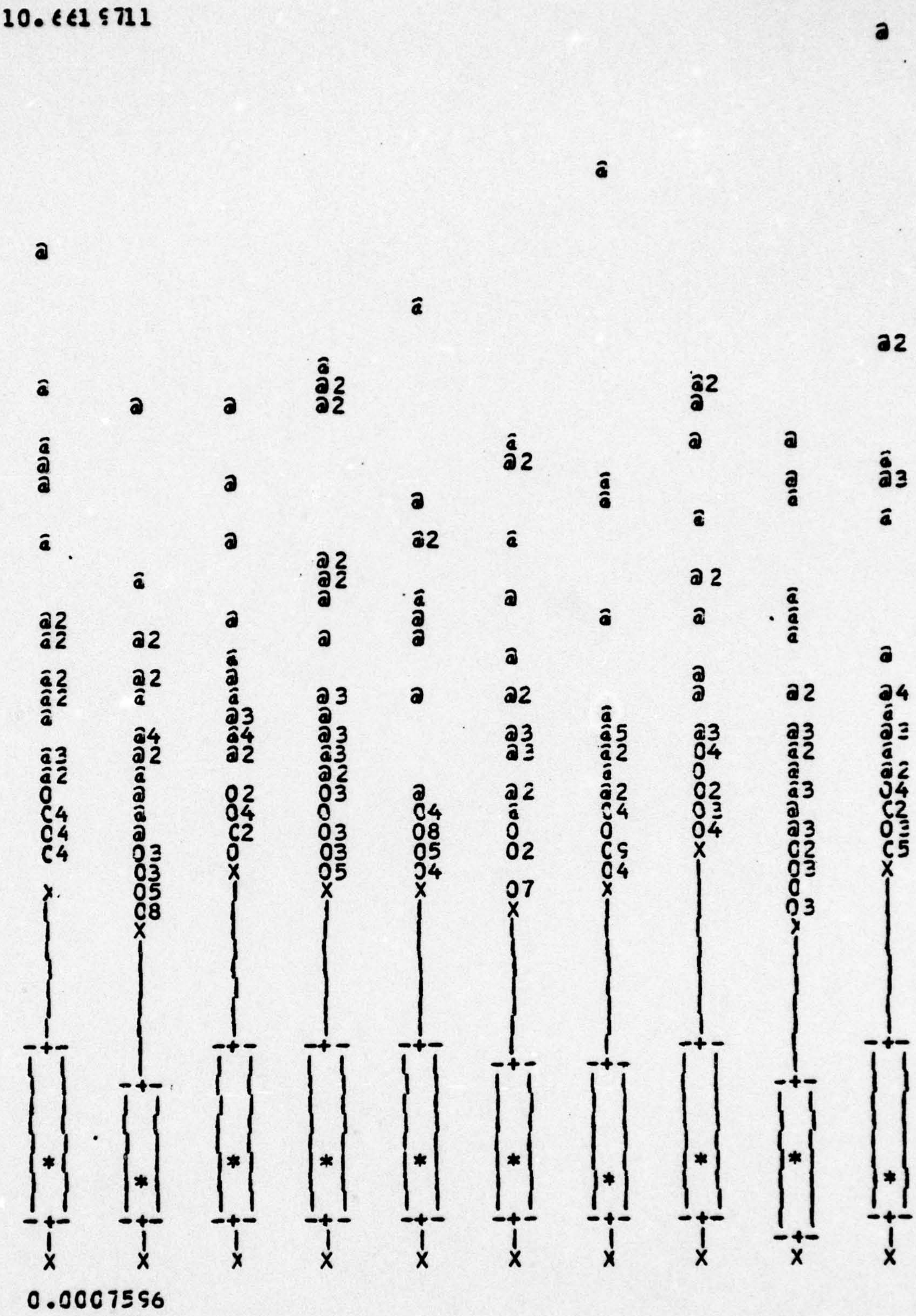


Figure 7g. Box plots of the sample waiting times $W_n(j)$, j=1,...,500 with zero waiting times removed for values of n = 61,000 to 70,000 in steps of 1,000. Correlated queue; t = 0.75, β = 0.5, ρ = 0.75. Figures at top and bottom at left are maximum and minimum sample values.

Another question of interest is how the parameters ρ and t affect the waiting time in the MD x MD/1 queue, and how these distributions differ from the exponential distribution of the positive values of W_n for the uncorrelated M/M/1 queue. There are several graphical techniques for doing this, and some of the results are as follows.

a) Box plots are used to compare the distribution of waiting times at $\beta = 0.5$ for several fixed values of traffic intensity t as the correlation parameter ρ is varied. These are shown in fig. 8a-d and complement Figure 3b also for fixed values of the correlation parameter ρ, the variation of the distribution of W is shown in Figs. 9a-f. Note that in these figures 10 approximately uncorrelated samples in the steady state have been combined.

b) The box plots are not useful for examining departures from exponentiality, and for this purpose the histograms and sample statistics from HIST F were utilized. For the grid of values of ρ and t discussed in (V.B) the coefficients of variation, skewness and kurtosis were extracted from the HISTF outputs (no zeros), and these are compared to the exponential values $C(k) = 1.0$, $\gamma_1 = 2.00$, $\gamma_2 = 6.00$ in table 4. It is clear that for small ρ & t the distribution is less skewed than the exponential, but as ρ and t increase the distribution becomes positively skewed and is certainly non-exponential. Thus for $\rho = 0.9$ and $t = 0.9$ we have $C(x) = 1.1325$, $\gamma_1 = 2.3289$, $\gamma_2 = 8.6449$. One explanation for this is that for

		Traffic Intensity				
		0.25	0.50	0.75	0.90	0.99
Correlation	0.0	p_1 = -0.0476	-0.1552	-0.2045	-0.1273	—
		p_2 = -0.2442	-0.5133	-0.5062	+0.2455	
		p_3 = -2.0026	-2.8502	-1.9100	+4.6589	
	0.25	p_1 = -0.0433	-0.1219	-0.1743	-0.1448	-0.0603
		p_2 = -0.2028	-0.4925	-0.5689	-0.3550	-0.3000
		p_3 = -1.7623	-3.0085	-3.0471	-2.1357	-2.223
	0.50	p_1 = -0.0132	-0.0303	-0.1117	-0.0839	—
		p_2 = -0.1963	-0.1725	-0.3894	-0.1937	
		p_3 = -1.8133	-1.5305	-2.0960	-1.3652	
	0.75	p_1 = +0.1625	+0.1191	+0.0605	+0.0364	—
		p_2 = +0.8135	+0.3999	-0.0189	+0.2571	
		p_3 = +6.6253	+1.9612	-0.8715	+2.3012	
	0.90	p_1 = +0.5039	+0.5023	+0.3245	+0.1325	—
		p_2 = +4.7702	+1.6374	+0.7767	+0.3289	
		p_3 =+71.3302	+12.8466	+5.8286	+2.6449	

p_1 = coefficient of variation - 1.0

p_2 = skewness - 2.0

p_3 = kurtosis - 6.0

Table 4. Deviations of the Estimated Sample Coefficients of Variation, Skewness and Kurtosis for the Correlated Queue Waiting Times (Without Zeros) from the Known (Exponential) Values for the M/M/1 Queue. For various ρ and t values.

ρ small, the service time is decreased by the correlation when many people arrive. The distribution is therefore tighter than for the M/M/1 case. However as ρ increases the service times become highly correlated through the cross-coupling, and this effect, which increases the mean waiting time and the skewness of the distribution, is dominant.

c) Another question of interest is whether for large t the heavy traffic approximation holds (Jacobs, 1978) and the distribution of W is approximately exponential. This is difficult to examine because the time for the queue to reach steady state is inordinately high for large t. Thus one case, $t = 0.995$, $\rho = 0.25$ was studied and the HISTF outputs are shown in Figures 10a & 10b. The plots and the sample statistics still show a large amount of underdispersion relative to an exponential distribution. Thus convergence to an exponential distribution as $t \to 1$, if it occurs, must be very slow.

Plots for an extreme case, $\rho = 0$, $\beta = 0$, are shown for various values of t from 0.25 to 0.99 in Figures 11a-e.

d) The question of judging when the transient in the simulation has died out is very complex and difficult, just like any question of detecting a signal in noise. In particular it is to be stressed that the more graphic output one has, the better off one is. This is another reason for looking at the $W_n(j)$ samples as well as the $\overline{W}_n(j)$ samples.

In Figure 12a $\hat{W}_n$ is shown for the MD x MD/1 queue with $t = 0.99$, $\rho = 0.25$. The judgement was made, and published

in Table 1, that equilibrium was reached by the time of the 40,000th customer and this appears reasonable from the figure, which goes out to n = 280,000. The plot of $\bar{\bar{W}}_n$ is given in Fig. 12b; superposing it on Fig. 12a is a help to visualization. It of course has a larger transient than $\hat{W}_n$ because it drags in <u>all</u> the previous annual times, so that the plot of $\hat{W}_n$ gives a better picture of the real steady state. Of course, as in any real simulation, E(W) is unknown and the asymptote in Fig. 12b is a help in showing the convergence. Note too that the normality in $\bar{\bar{W}}_n$ decreases with n, but that of $\hat{W}_n$ does not.

Note too that since we have determined that W_n's 1,000 arrivals apart are almost uncorrelated, local smoothing could be applied to the $\hat{W}_n$ plot to obtain a better picture.

Output from a similar simulation for an M/M/1 queue, again with t = 0.99 and 500 replications, is shown in Figs. 12c & d. The mean of W is known to be E(W) = 24.75. However the plot of $\hat{W}_n$ is well above this from approximately n = 60,000 to n = 260,000, and since the correlation between $\hat{W}_n$ values is not very extensive, this probably indicates that the transient is such that $E(W_n)$ does not go up monotonically to E(W), as one might expect intuitively or from the simulation out to n = 240,000, but actually rises above E(W) and then declines. It may even have a damped oscillation. The point is that <u>all</u> the data should be examined- formal tests of the similarity of the W_n samples at say $W_{100,000}$ and $W_{110,000}$ would probably have indicated convergence (i.e., similar distributions).

Another overlaying the curve of $\bar{\bar{W}}_n$ on that of $\bar{\bar{W}}_n$ is a help usually, as is the horizontal asymptotic in Fig. 12d. Note that the initial transient in the M/M/1 is longer than it is in the MD x MD/1 queue.

Another question which arises is whether one can start the simulation in such a fashion as to reduce the length of the transient. An attempt to do this is shown in Figs. 13a-b. Here an extreme case, t = 0.995, was taken and the initial value W_o was not taken to be zero, but an exponential random variable with mean 29.0. The value 29.0 comes from the prediction formula IV.1. Roughly speaking there appears to be faster convergence. The simulation starting from $W_o = 0$ takes a long time to converge and was too extensive to be handled on the computer.

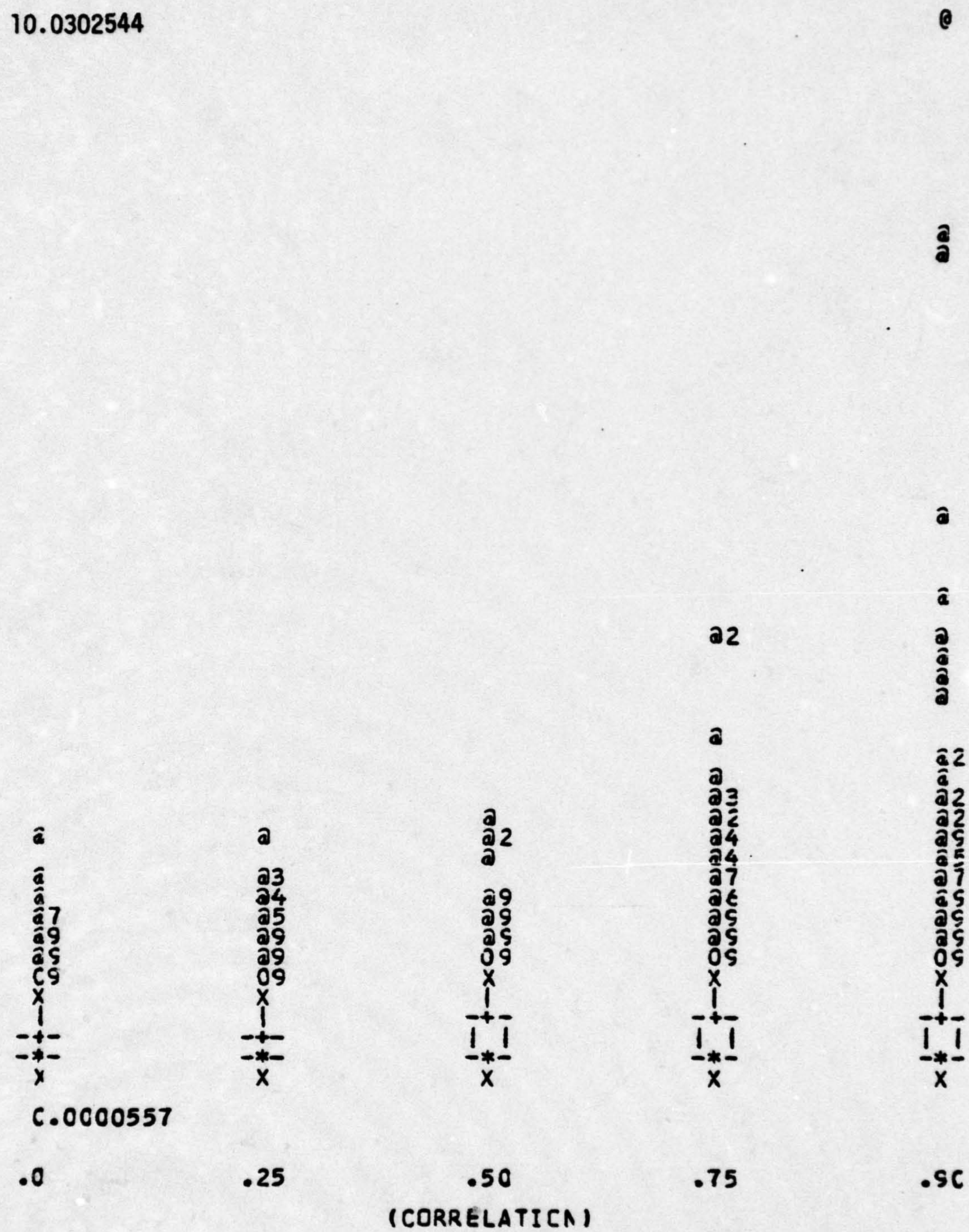


Figure 8a. Correlated queue. Box plots of a sample of waiting times $W_{n_i}(j)$, $j=1,\ldots,500$; $i=1,\ldots,10$; with zero waiting times removed, for $t = 0.25$ and five values of ρ. For each (t,ρ) pair n_1 is chosen to be large enough for the queue to be in steady state, and 10 approximately uncorrelated samples at $n_1, n_2, \ldots, n_{10}$ are combined.

Here n_1 = 31,000, 31,000, 31,000, 31,000, 51,000

as ρ = 0.0, 0.25, 0.50, 0.75, 0.90, respectively. $\beta = 0.5$.

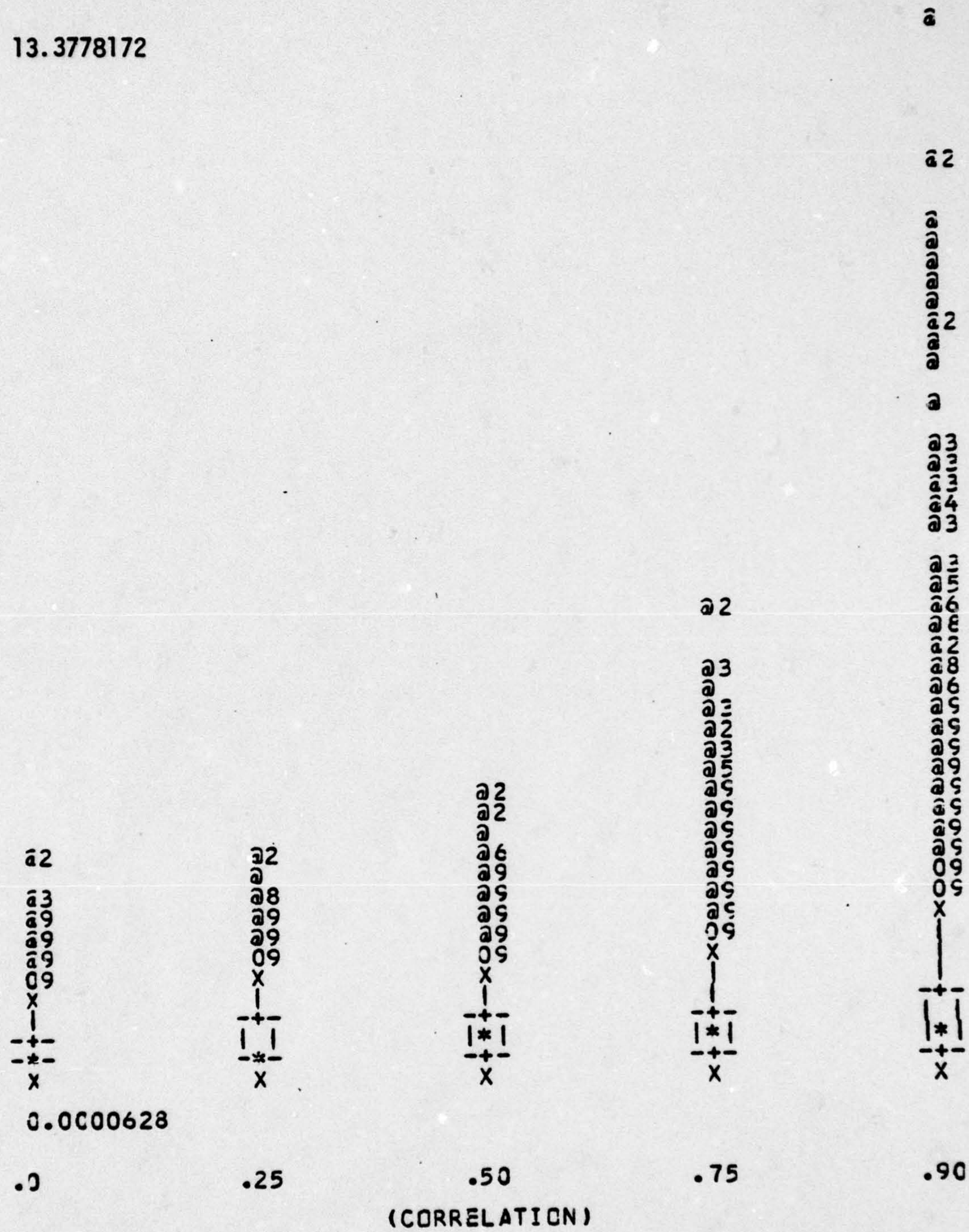


Figure 8b. Correlated queue. Box plots of a sample of waiting times $W_{n_i}(j)$, j=1,...,500; i=1,...,10; with zero waiting times removed, for t = 0.50 and five values of ρ. For each (t,ρ) pair n_1 is chosen to be large enough for the queue to be in steady state, and 10 approximately uncorrelated samples at $n_1, n_2, \ldots, n_{10}$ are combined.
Here n_1 = 31,000, 41,000, 41,000, 41,000, 61,000 as ρ = 0.0, 0.25, 0.50, 0.75, 0.90, respectively. β = 0.5.

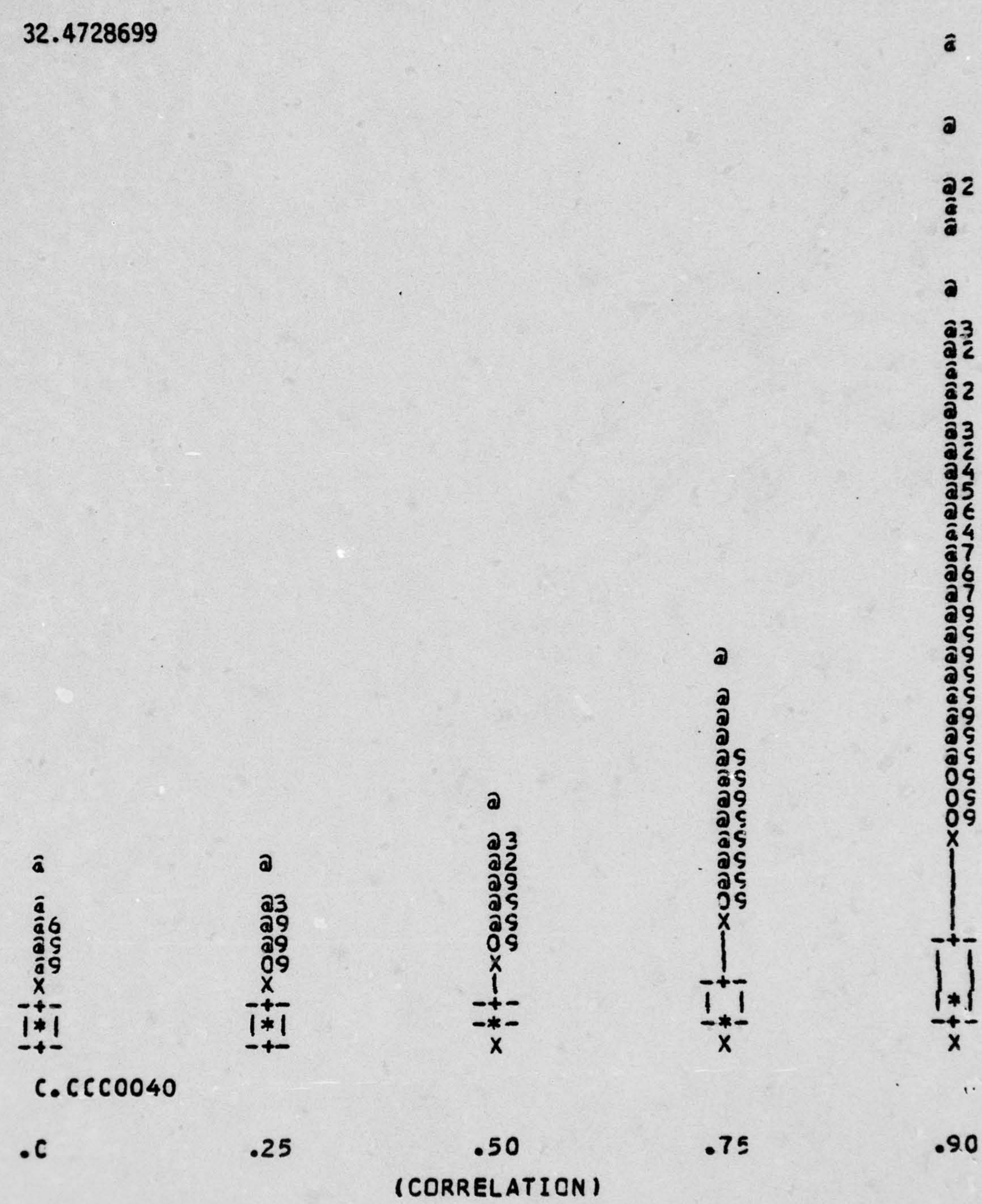


Figure 8c. Correlated queue. Box plots of a sample of waiting times $W_{n_i}(j)$, $j=1,\ldots,500$; $i=1,\ldots,10$; with zero waiting times removed, for $t = 0.75$ and five values of ρ. For each (t,ρ) pair n_1 is chosen to be large enough for the queue to be in steady state, and 10 approximately uncorrelated samples at $n_1, n_2, \ldots, n_{10}$ are combined.

Here n_1 = 41,000, 41,000, 41,000, 61,000, 91,000 as ρ = 0.0, 0.25, 0.50, 0.75, 0.90, respectively. $\beta = 0.5$.

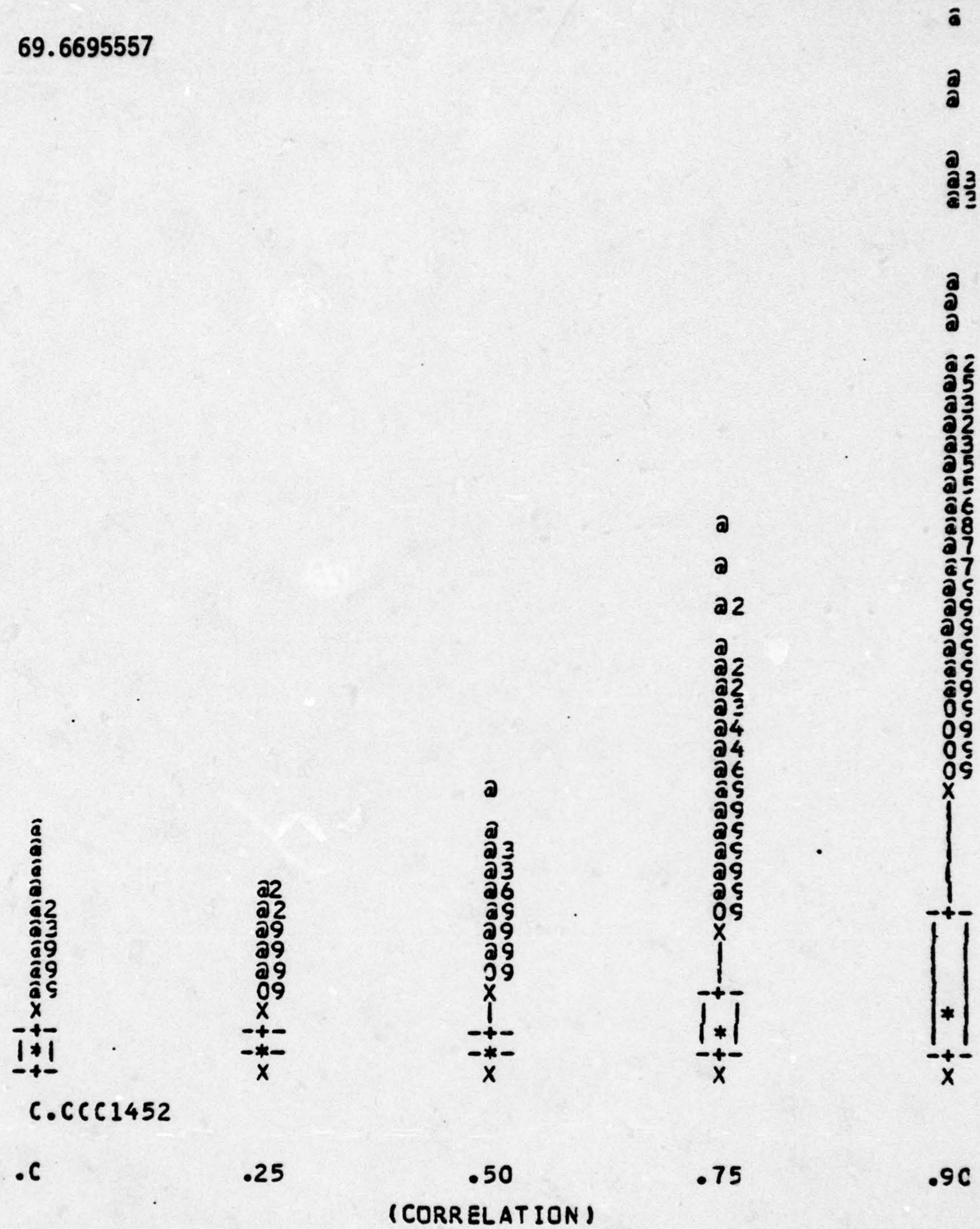


Figure 8d. Correlated queue. Box plots of a sample of waiting times $W_{n_i}(j)$, $j=1,\ldots,500$; $i=1,\ldots,10$; with zero waiting times removed, for $t = 0.90$ and five values of ρ. For each (t,ρ) pair n_1 is chosen to be large enough for the queue to be in steady state, and 10 approximately uncorrelated samples at $n_1, n_2, \ldots, n_{10}$ are combined.

Here n_1 = 51,000, 61,000, 61,000, 91,000, 91,000

as ρ = 0.0, 0.25, 0.50, 0.75, 0.90, respectively. $\beta = 0.5$.

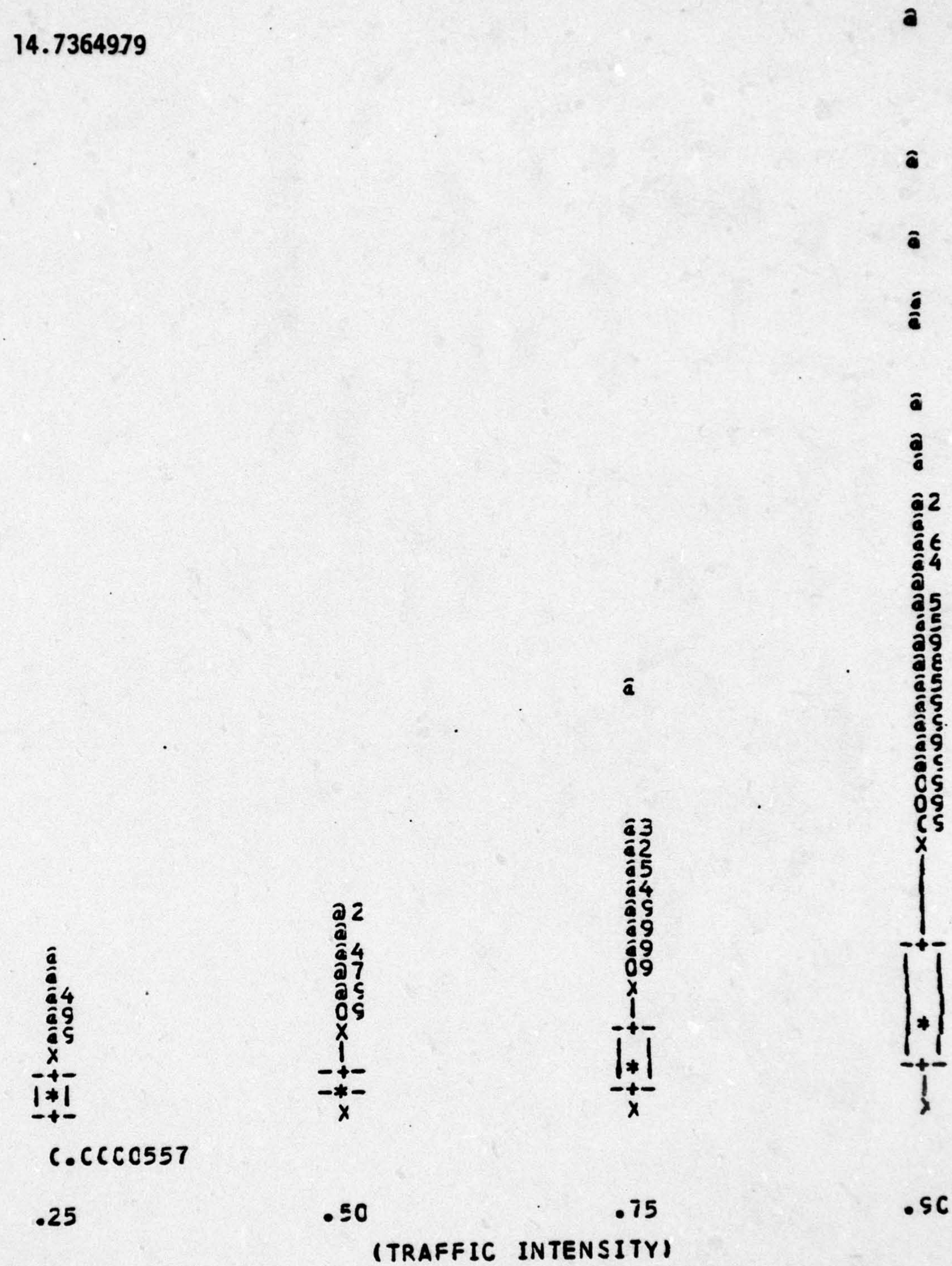


Figure 9a. Correlated queue. Box plots of a sample of waiting times $W_{n_i}(j)$, j=1,...,500; i=1,...,10; with zero waiting times removed, for $\rho = 0.0$ and four values of t. For each (t,ρ) pair n_1 is chosen to be large enough for the queue to be in steady state, and 10 approximately uncorrelated samples at $n_1, n_2, \ldots, n_{10}$ are combined. $\beta = 0.5$.

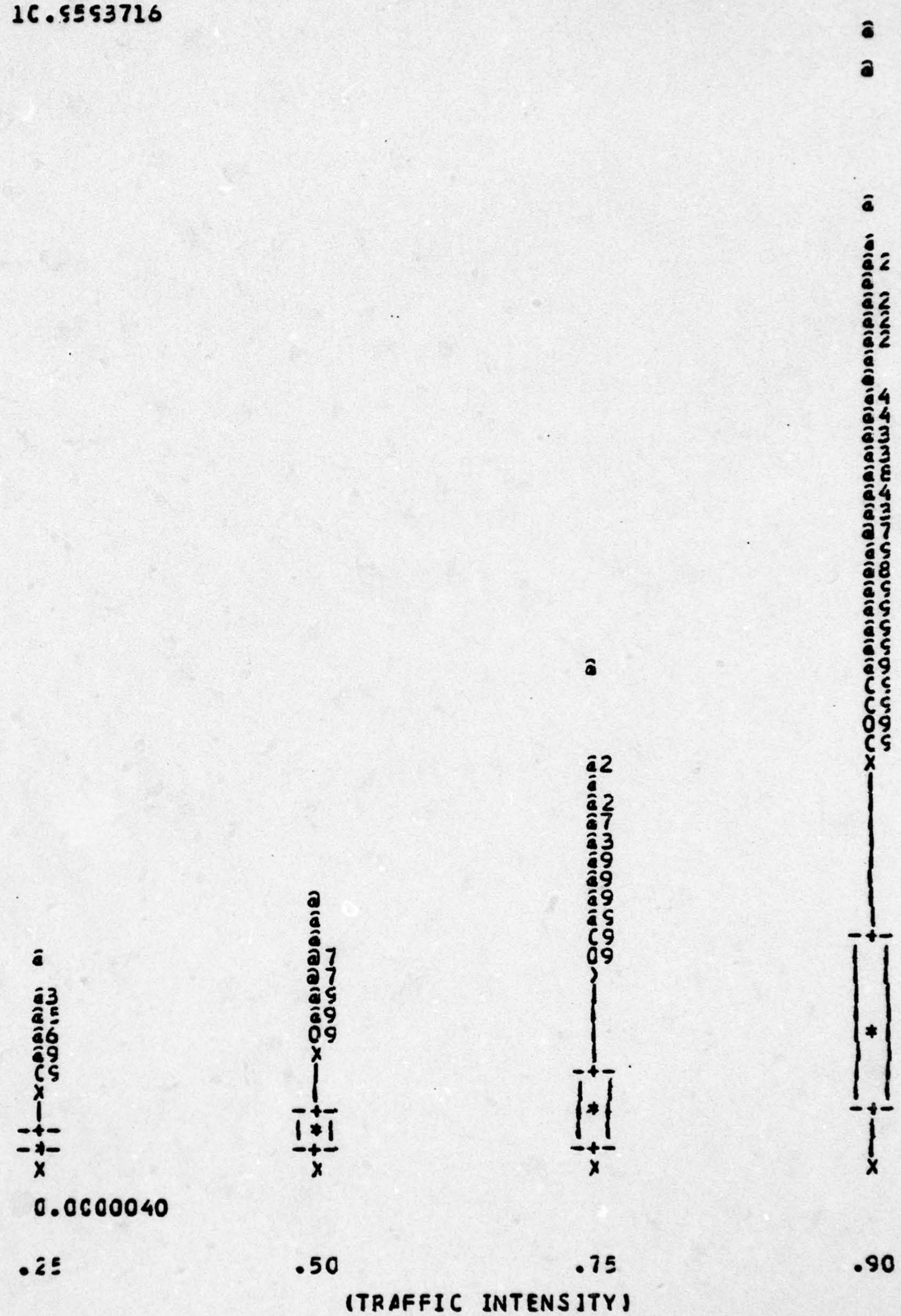


Figure 9b. Correlated queue. Box plots of a sample of waiting times $W_{n_i}(j)$, $j=1,\ldots,500$; $i=1,\ldots,10$; with zero waiting times removed, for $\rho = 0.25$ and four values of t. For each (t,ρ) pair n_1 is chosen to be large enough for the queue to be in steady state, and 10 approximately uncorrelated samples at $n_1, n_2, \ldots, n_{10}$ are combined. $\beta = 0.5$.

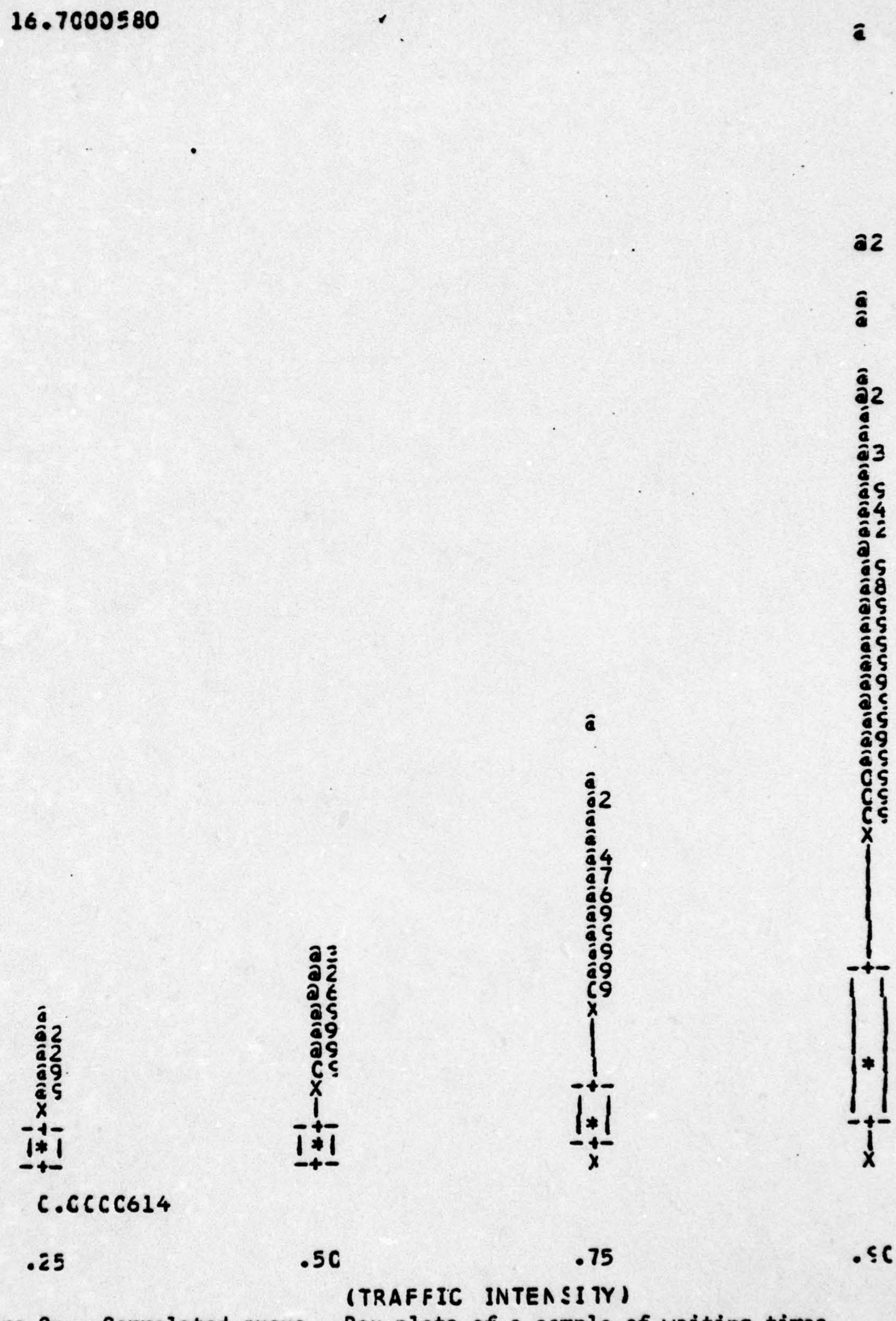


Figure 9c. Correlated queue. Box plots of a sample of waiting times $W_{n_i}(j)$, $j=1,\ldots,500$; $i=1,\ldots,10$; with zero waiting times removed, for $\rho = 0.50$ and four values of t. For each (t,ρ) pair n_1 is chosen to be large enough for the queue to be in steady state, and 10 approximately uncorrelated samples at $n_1, n_2, \ldots, n_{10}$ are combined. $\beta = 0.5$.

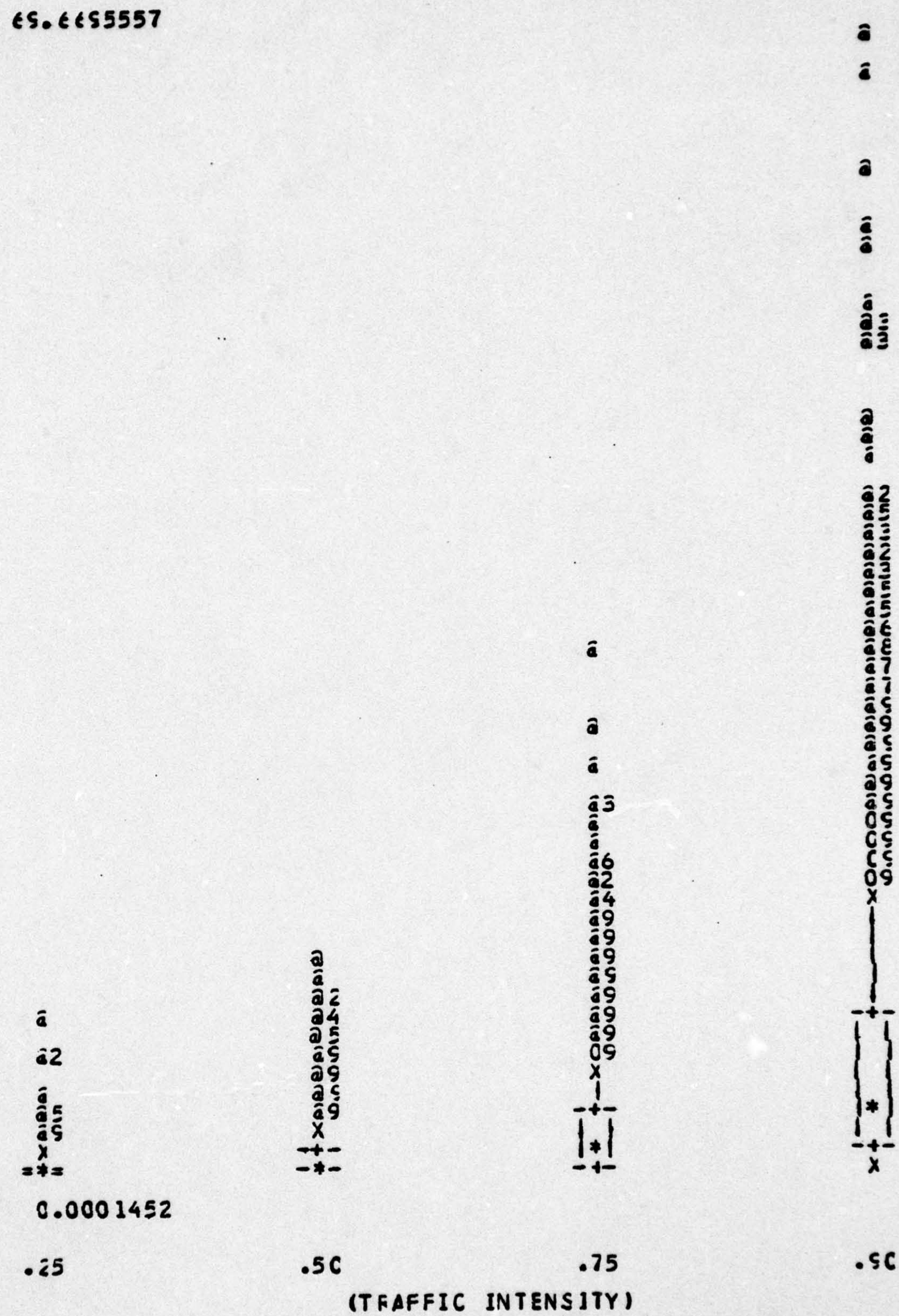
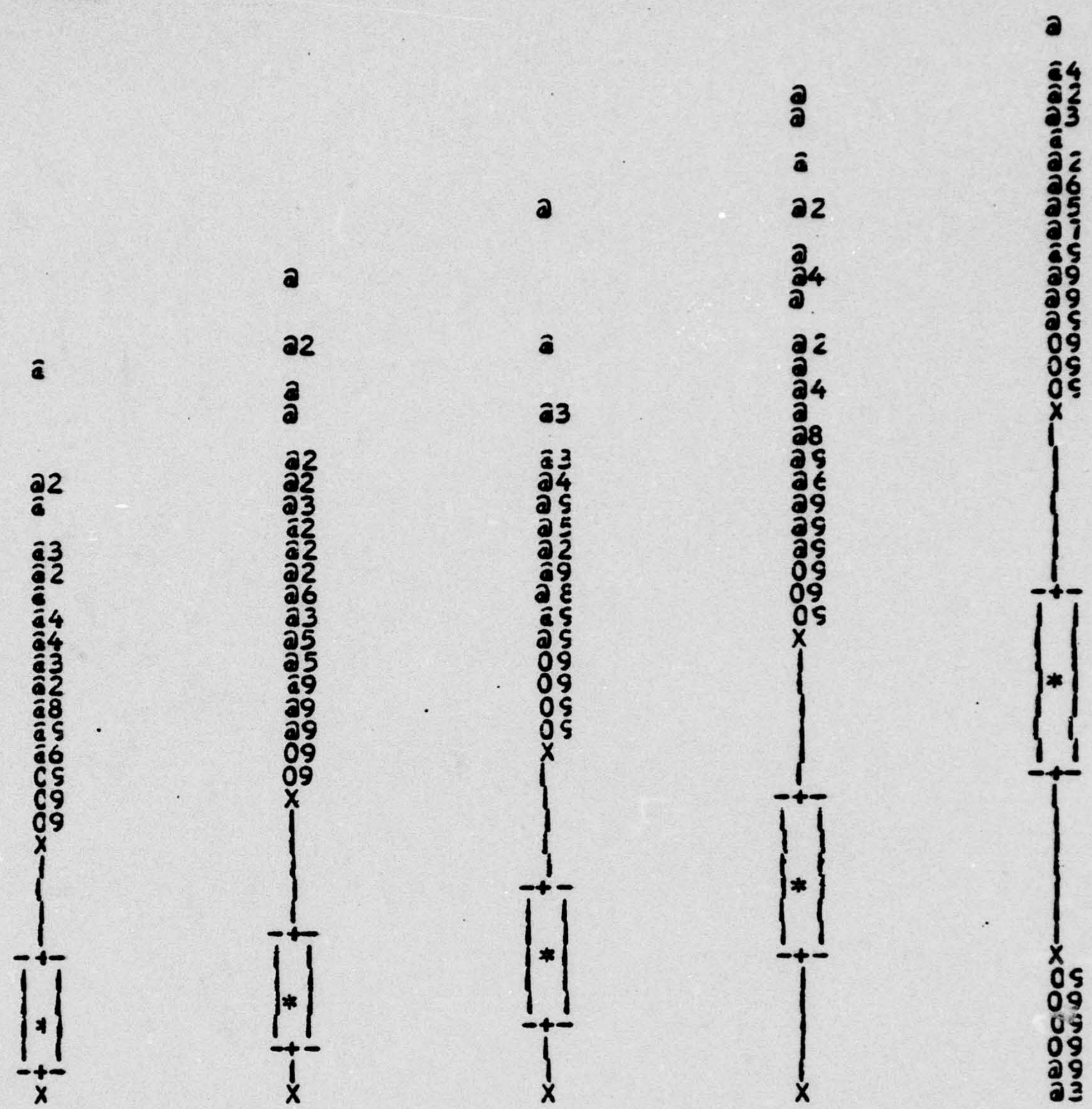


Figure 9e. Correlated queue. Box plots of a sample of waiting times $W_{n_i}(j)$, $j=1,\ldots,500$; $i=1,\ldots,10$; with zero waiting times removed, for $\rho = 0.90$ and four values of t. For each (t,ρ) pair n_1 is chosen to be large enough for the queue to be in steady state, and 10 approximately uncorrelated samples at $n_1, n_2, \ldots, n_{10}$ are combined. $\beta = 0.5$.

3.4176884



C.CCC0060

.25 .50 .75 .90 .99

(TRAFFIC INTENSITY)

Figure 9f. Correlated queue, $\rho = 0.0$, $\beta = 0.0$ (service time equals t times previous inter-arrival time). Box plots of a sample of waiting times $W_{n_i}(j)$, $j=1,\ldots,500$; $i=1,\ldots,10$; with zero waiting times removed. Five values of t. For each t, n_1 is chosen to be large enough for the queue to be in steady state, and 10 approximately uncorrelated samples at $n_1, n_2, \ldots, n_{10}$ are combined.

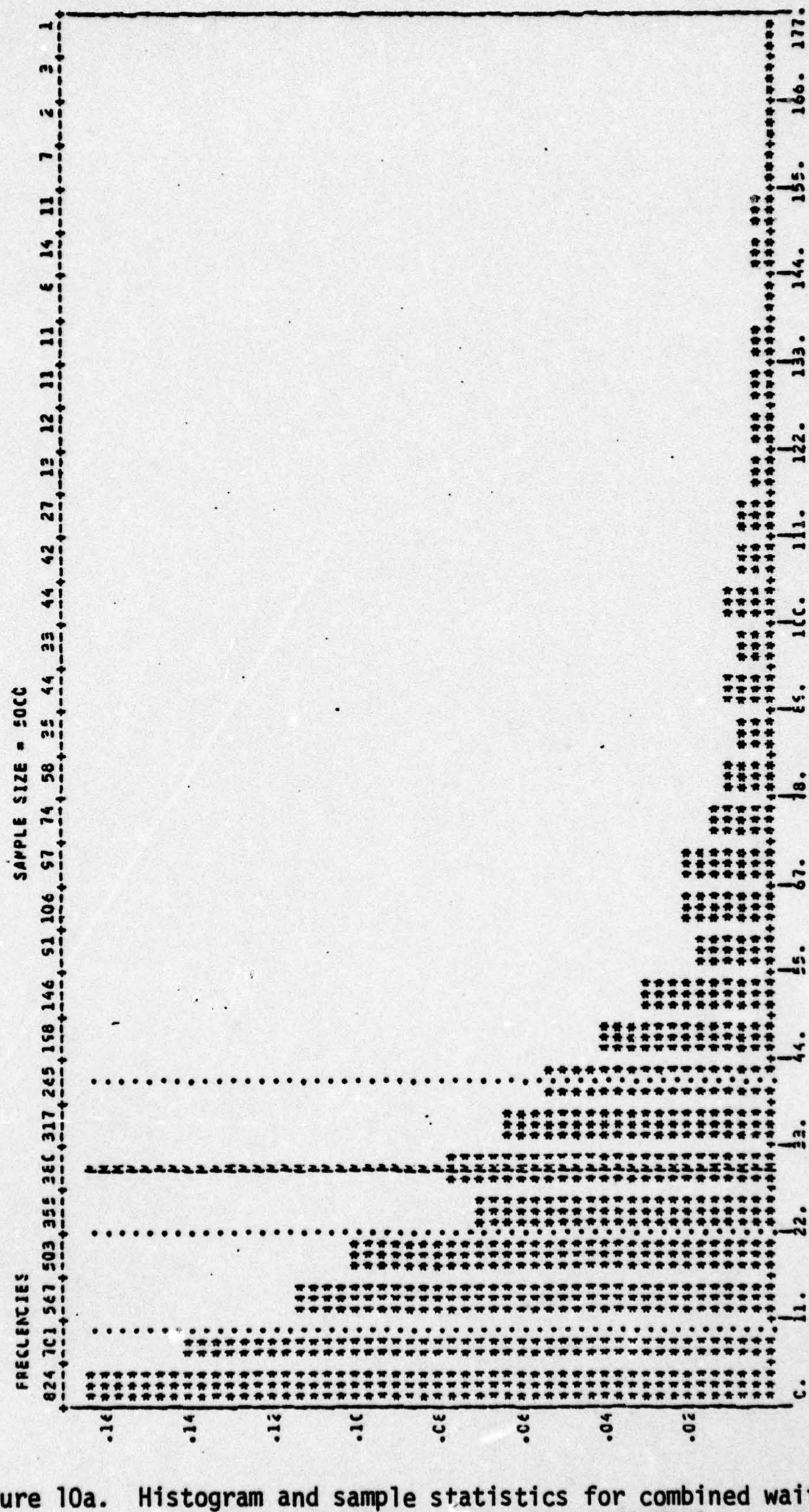


Figure 10a. Histogram and sample statistics for combined waiting-time data in the simulated, correlated MDXMD/1 queue; $t = 0.995$, $\beta = 0.5$, $\rho = 0.25$, $1/E(S) = 4.0$, i.e. all $W_n(j)$'s for $j=1,\ldots,500$ and $n = 91{,}000(1000)100{,}000$.

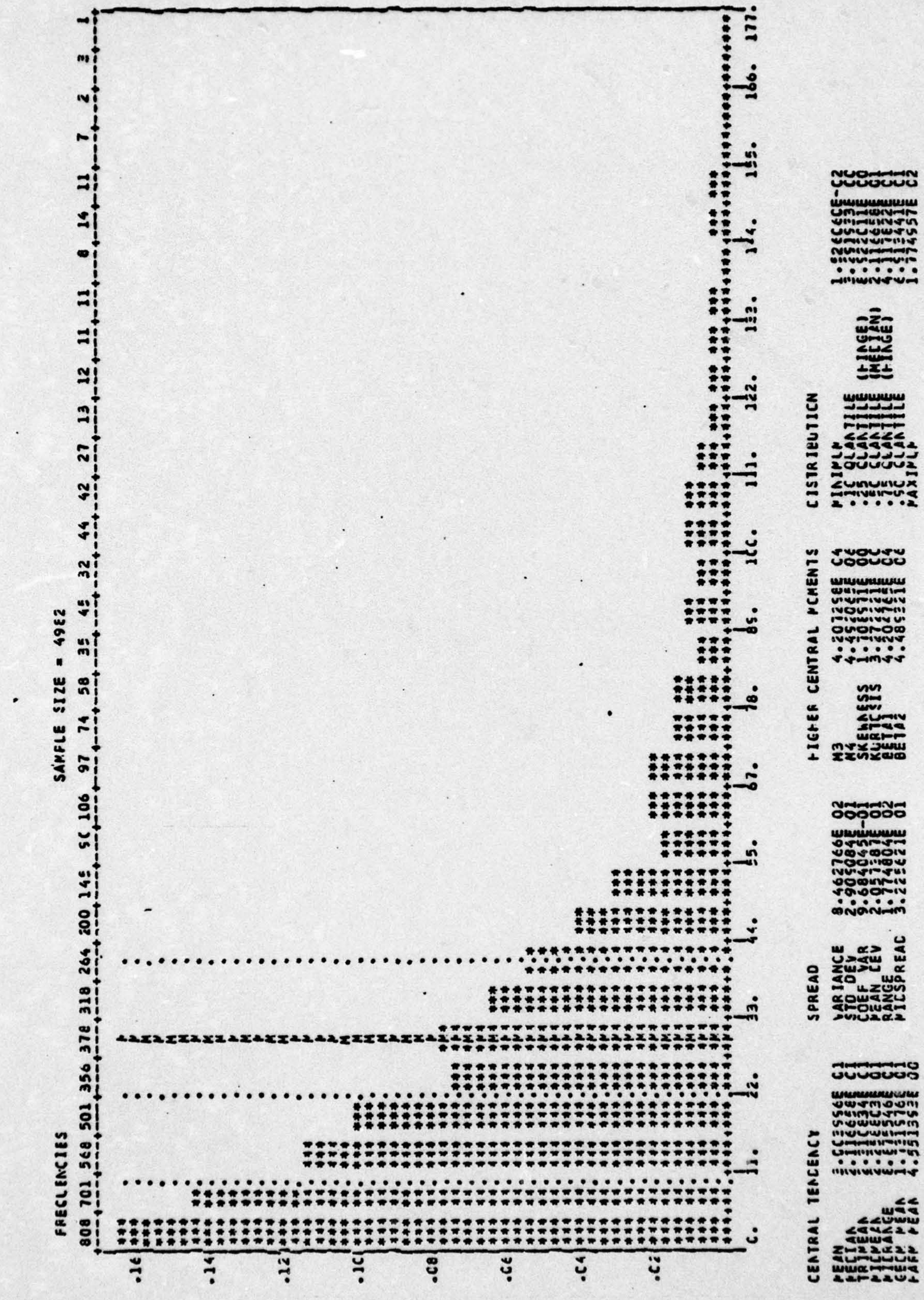


Figure 10b. Histogram and sample statistics for combined waiting-time data (with 18 zero waiting times removed) in the simulated, correlated MDXMD/1 queue; $t = 0.995$, $\beta = 0.5$, $\rho = 0.25$, $1/E(S) = 4.0$, i.e. all $W_n(j)$'s for $j = 1,\ldots,500$ and $n = 91{,}000(1000)100{,}000$.

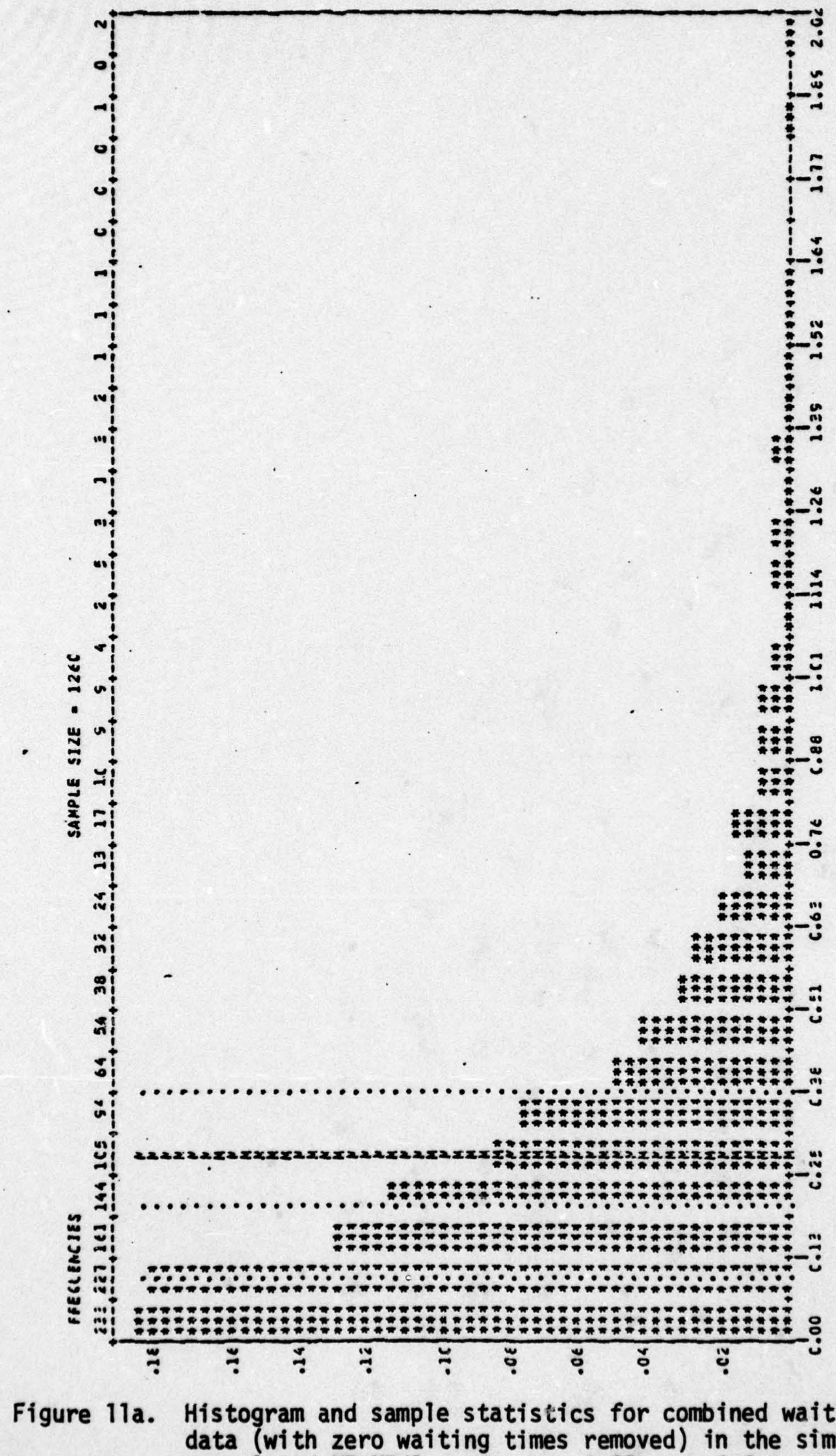


Figure 11a. Histogram and sample statistics for combined waiting-time data (with zero waiting times removed) in the simulated, correlated MDxMD/1 queue; t = 0.25, β = 0.0, ρ = 0.0, 1/E(S) = 4.0, i.e. all $W_n(j)$'s for j = 1,...,500 and n = 11,000(1000)20,000.

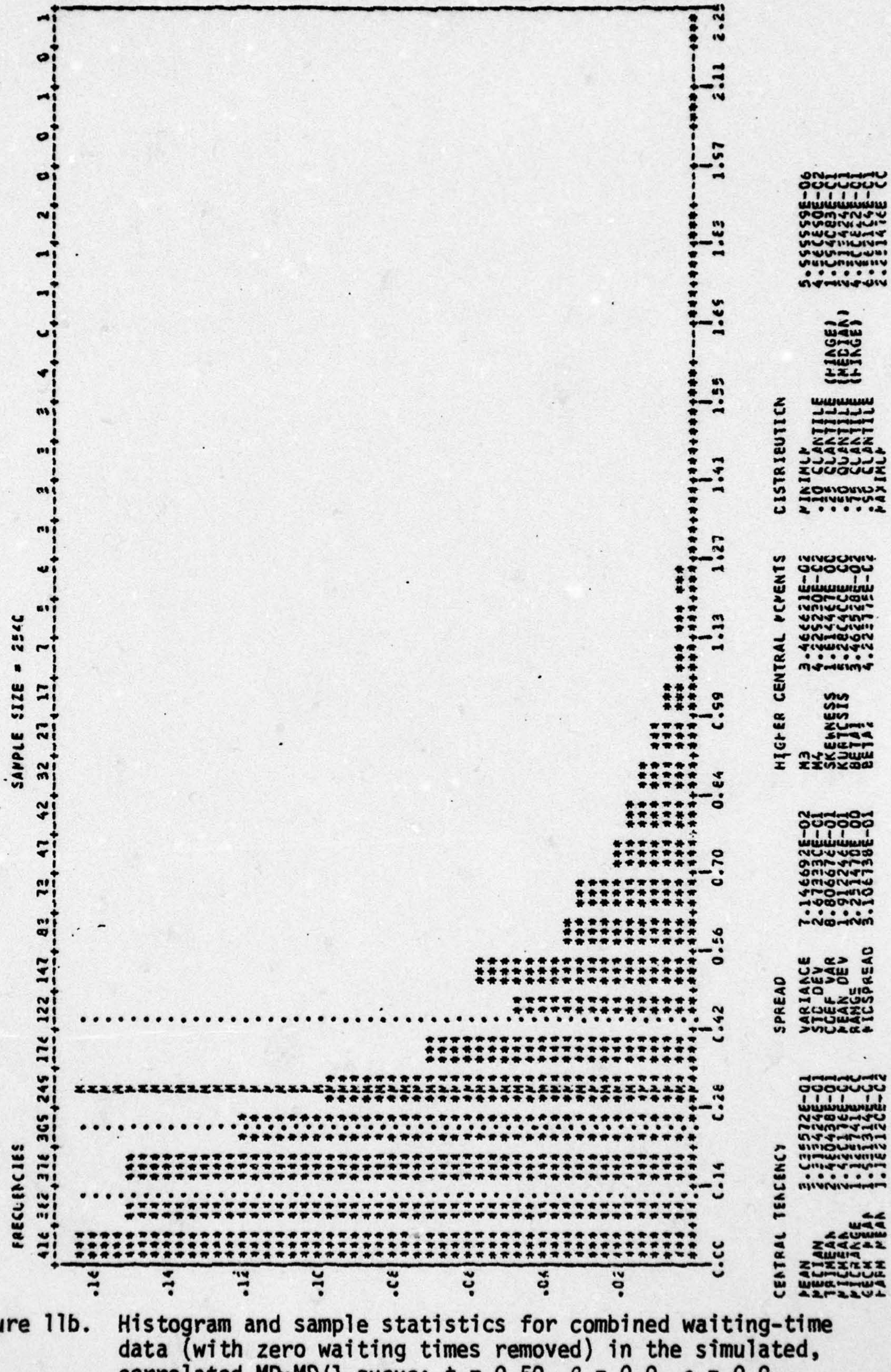


Figure 11b. Histogram and sample statistics for combined waiting-time data (with zero waiting times removed) in the simulated, correlated MDxMD/1 queue; $t = 0.50$, $\beta = 0.0$, $\rho = 0.0$, $1/E(S) = 4.0$, i.e. all $W_n(j)$'s for $j = 1,\ldots,500$ and $n = 21{,}000(1000)30{,}000$.

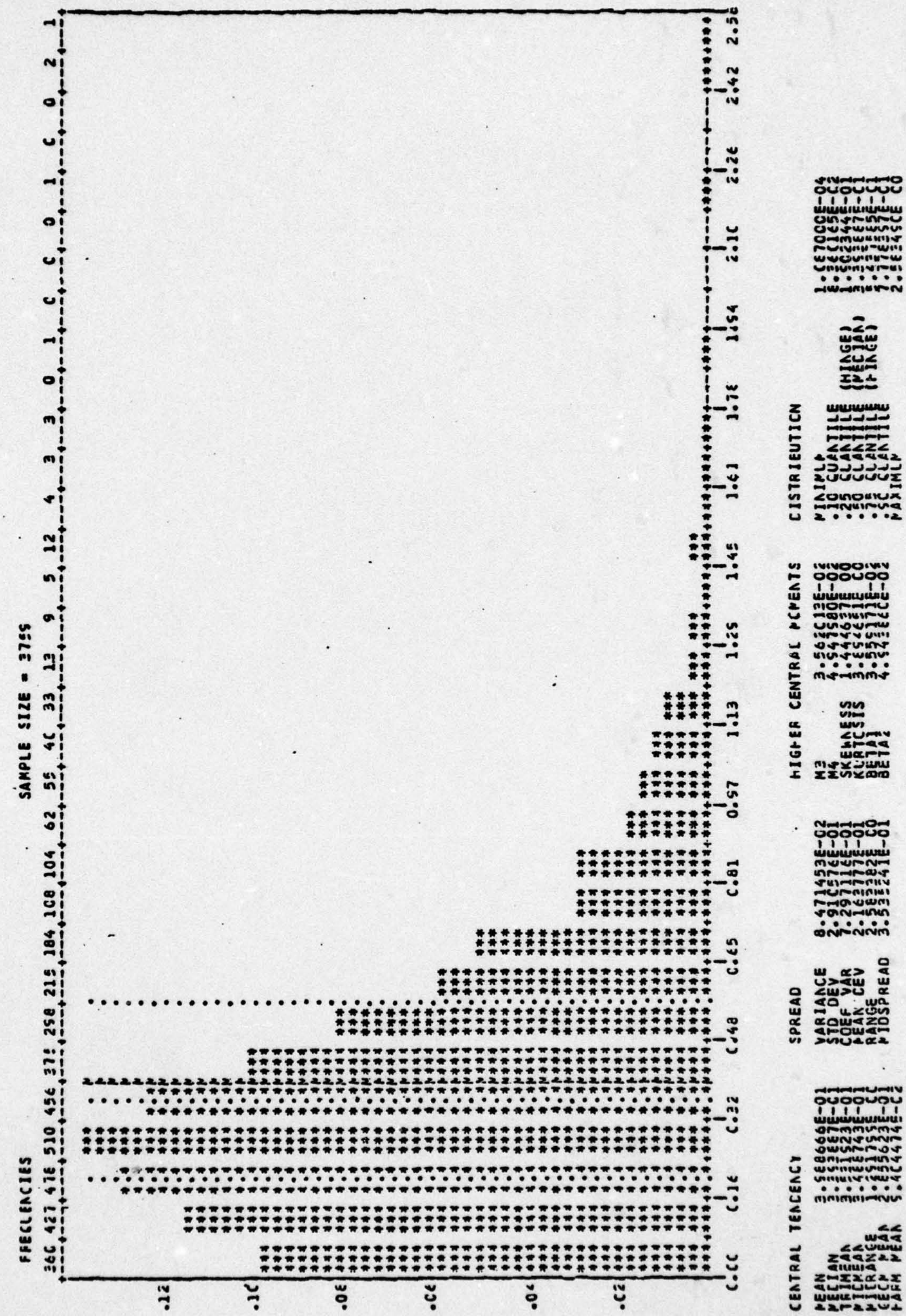


Figure 11c. Histogram and sample statistics for combined waiting-time data (with zero waiting times removed) in the simulated, correlated MDxMD/1 queue; $t = 0.75$, $\beta = 0.0$, $\rho = 0.0$, $1/E(S) = 4.0$, i.e. all $W_n(j)$'s for $j = 1,\ldots,500$ and $n = 21{,}000(1000)30{,}000$.

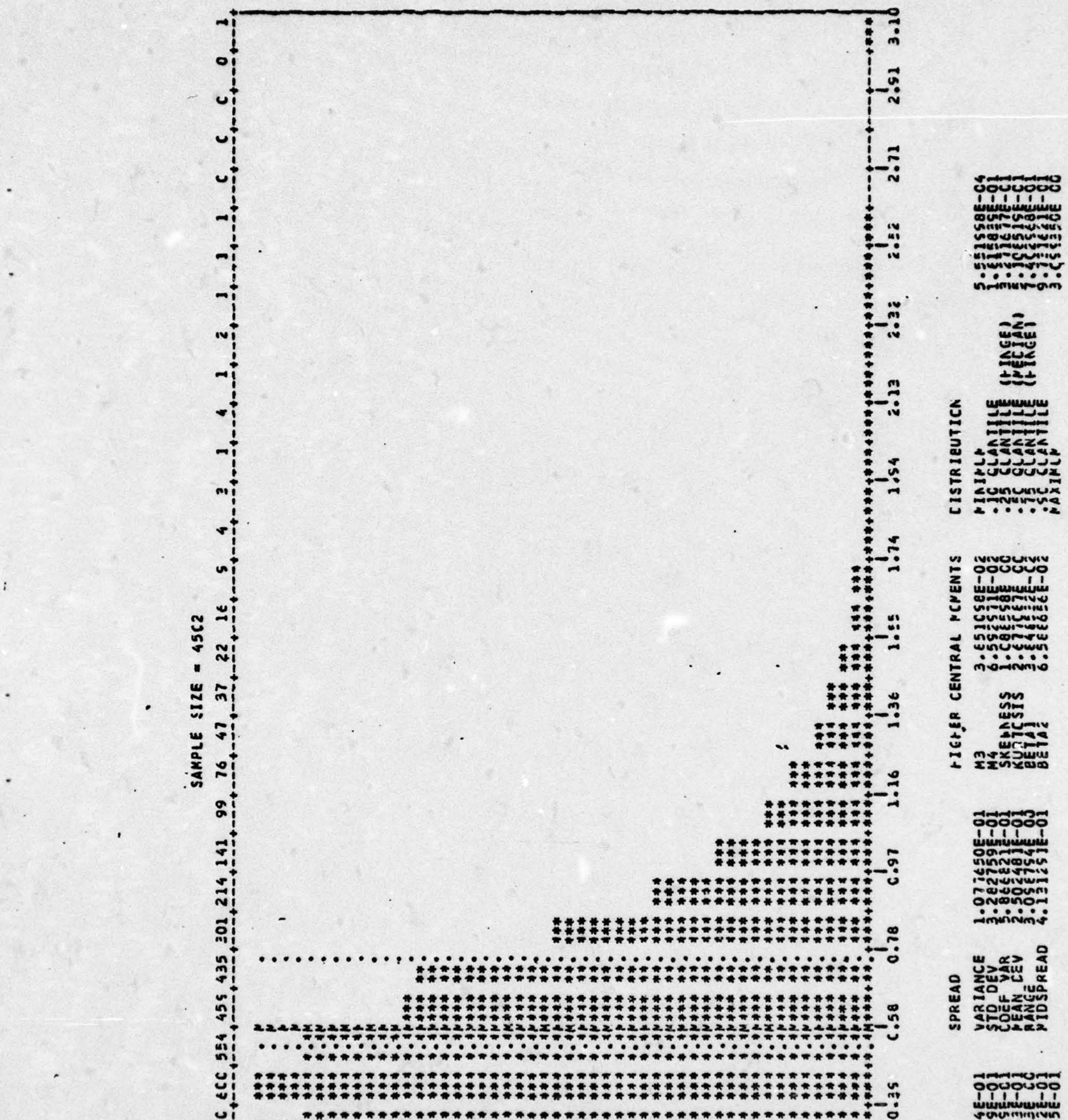


Figure 11d. Histogram and sample statistics for combined waiting-time data (with zero waiting times removed) in the simulated, correlated MDxMD/1 queue; t = 0.90, β = 0.0, ρ = 0.0, 1/E(S) = 4.0, i.e. all $W_n(j)$'s for j = 1,...,500 and n = 21,000(1000)30,000.

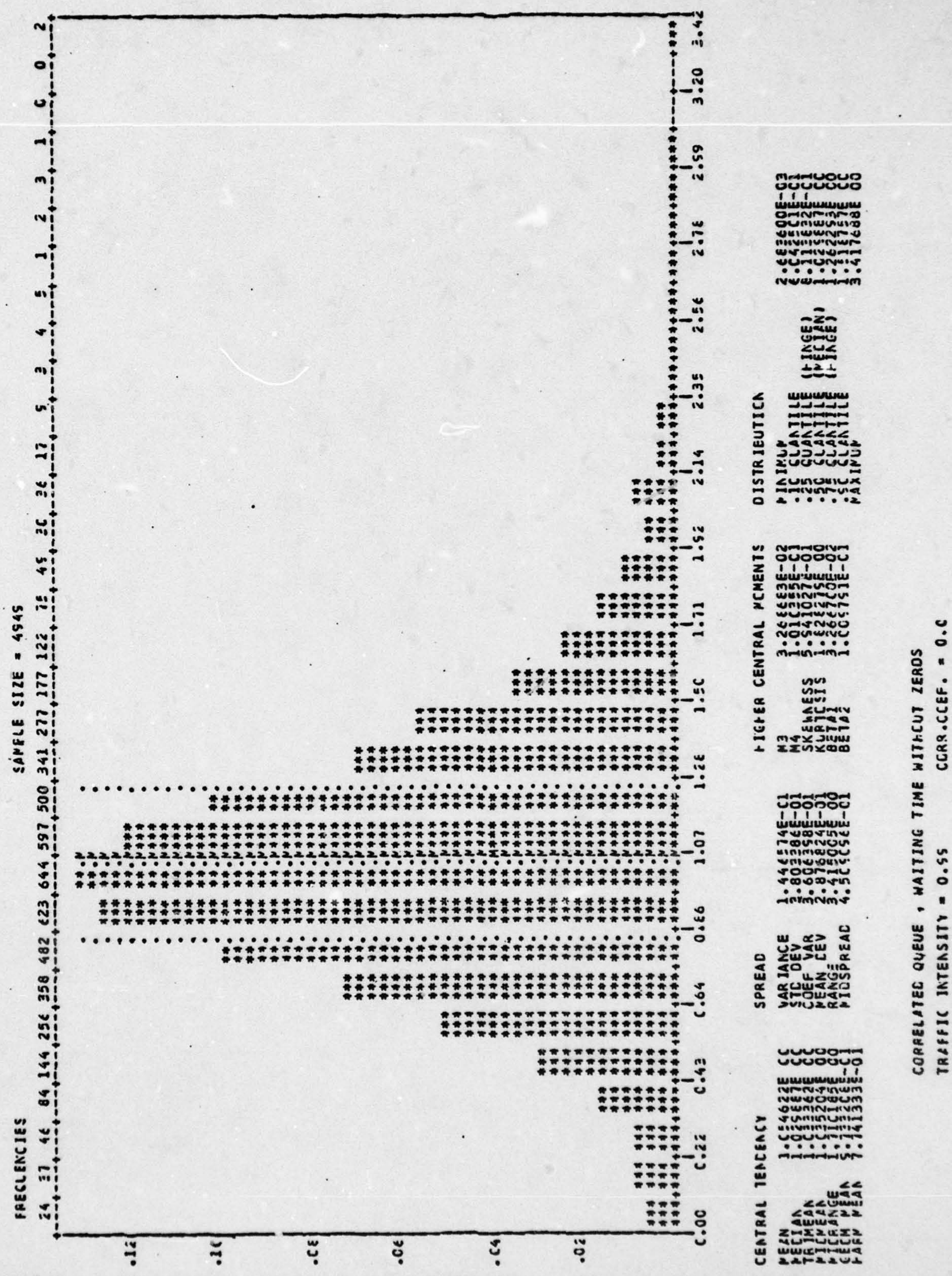


Figure 11e. Histogram and sample statistic for combined waiting-time data (with zero waiting times removed) in the simulated, correlated MDxMD/1 queue; t = 0.99, β = 0.0, ρ = 0.0, 1/E(S) = 4.0, i.e. all $W_n(j)$'s for j = 1,...,500 and n = 21,000(1000)30,000.

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A SIMULATION STUDY OF A CLASS OF FIRST-COME FIRST-SERVED QUEUES--ETC(U)

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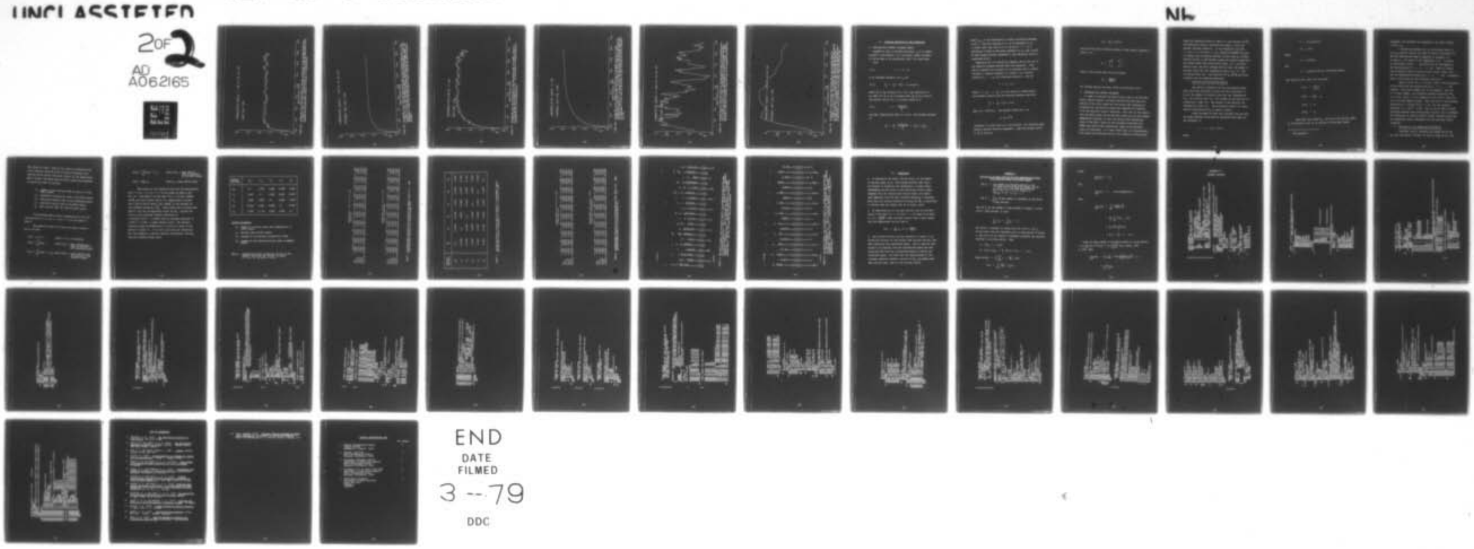

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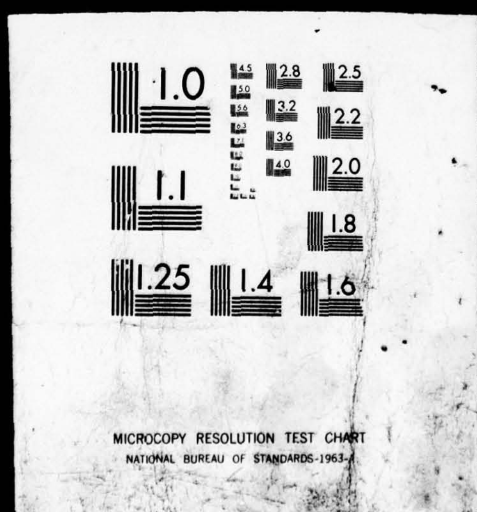

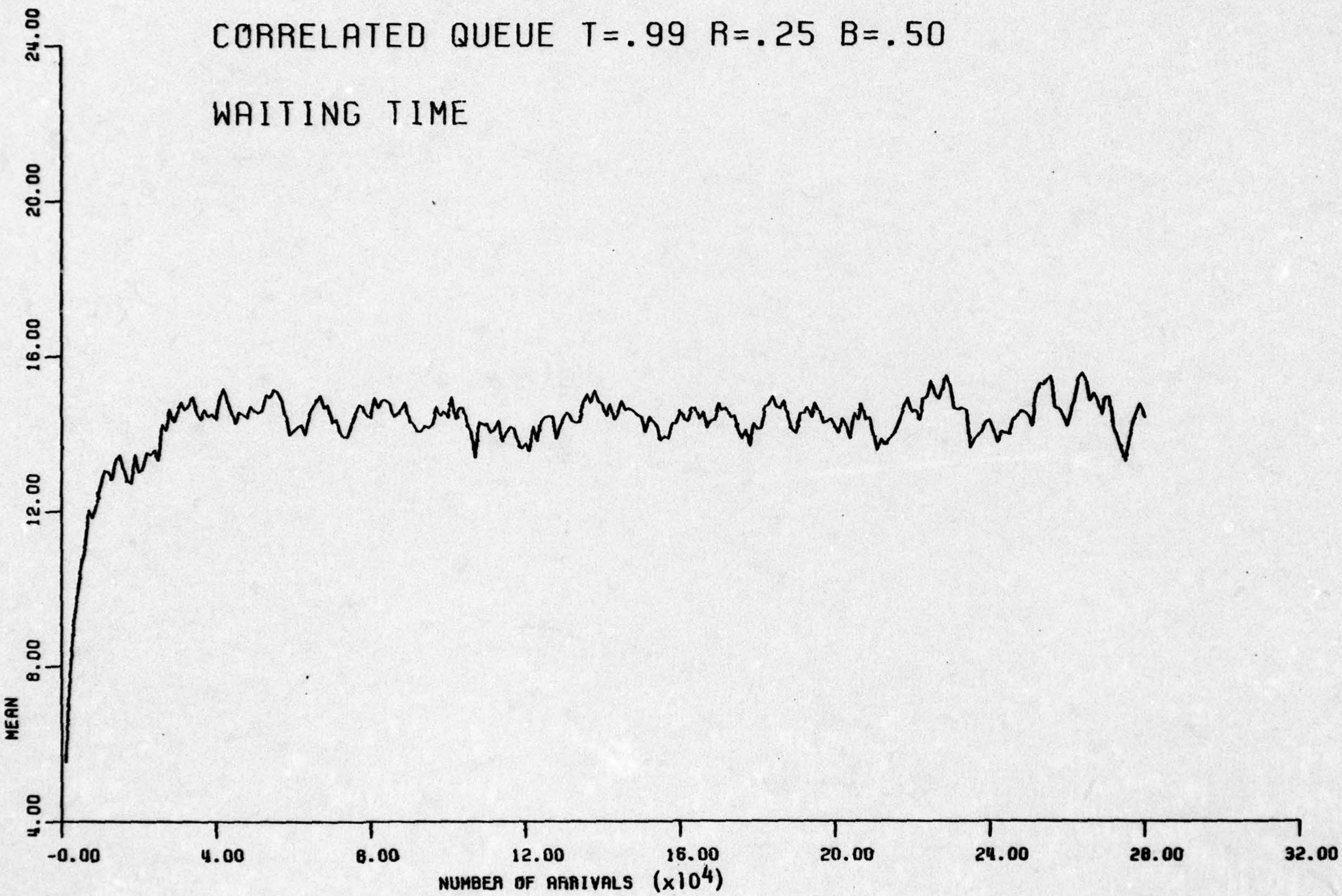

Figure 12a. Estimate $\hat{W}_n$ across 500 replications of the mean waiting time in the correlated queue for $t = 0.99$, $\rho = 0.25$, $\beta = 0.50$. The mean appears to fluctuate around the mean value (14.44) computed from the approximate formula (V.1) after $n = 40,000$ arrivals.

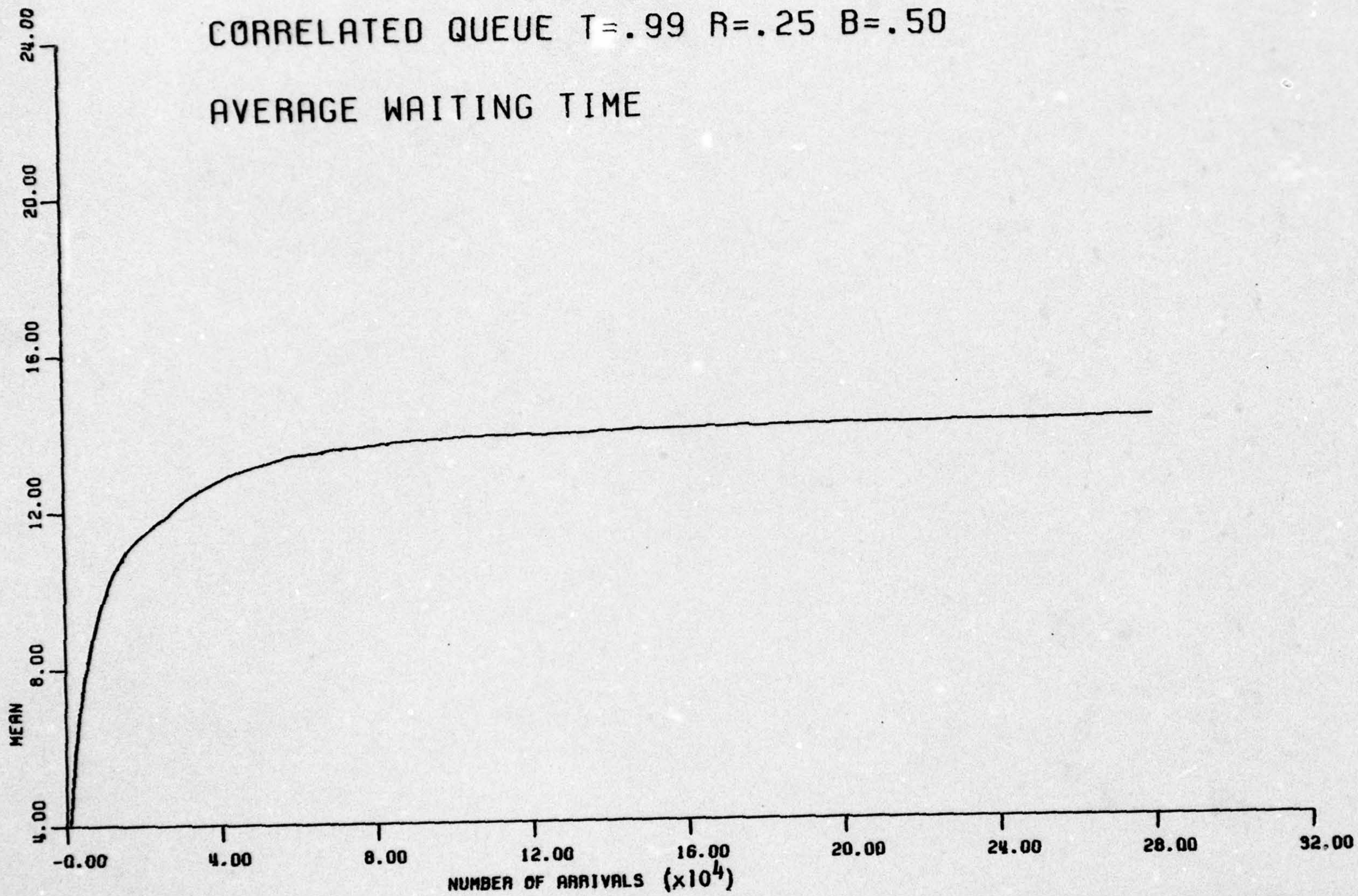

Figure 12b. Estimate $\overline{\overline{W}}_n$ of the stationary mean waiting time in the correlated queue for t = 0.99, $\rho = 0.25$, $\beta = 0.50$. This estimate has smaller sampling variance than $\hat{W}_n$ in Figure 12a because it is an average across 500 replications and along the 500 sample paths. The convergence is slower than for $\hat{W}_n$ because all $W_n(j)$, including the initial transient, are used.

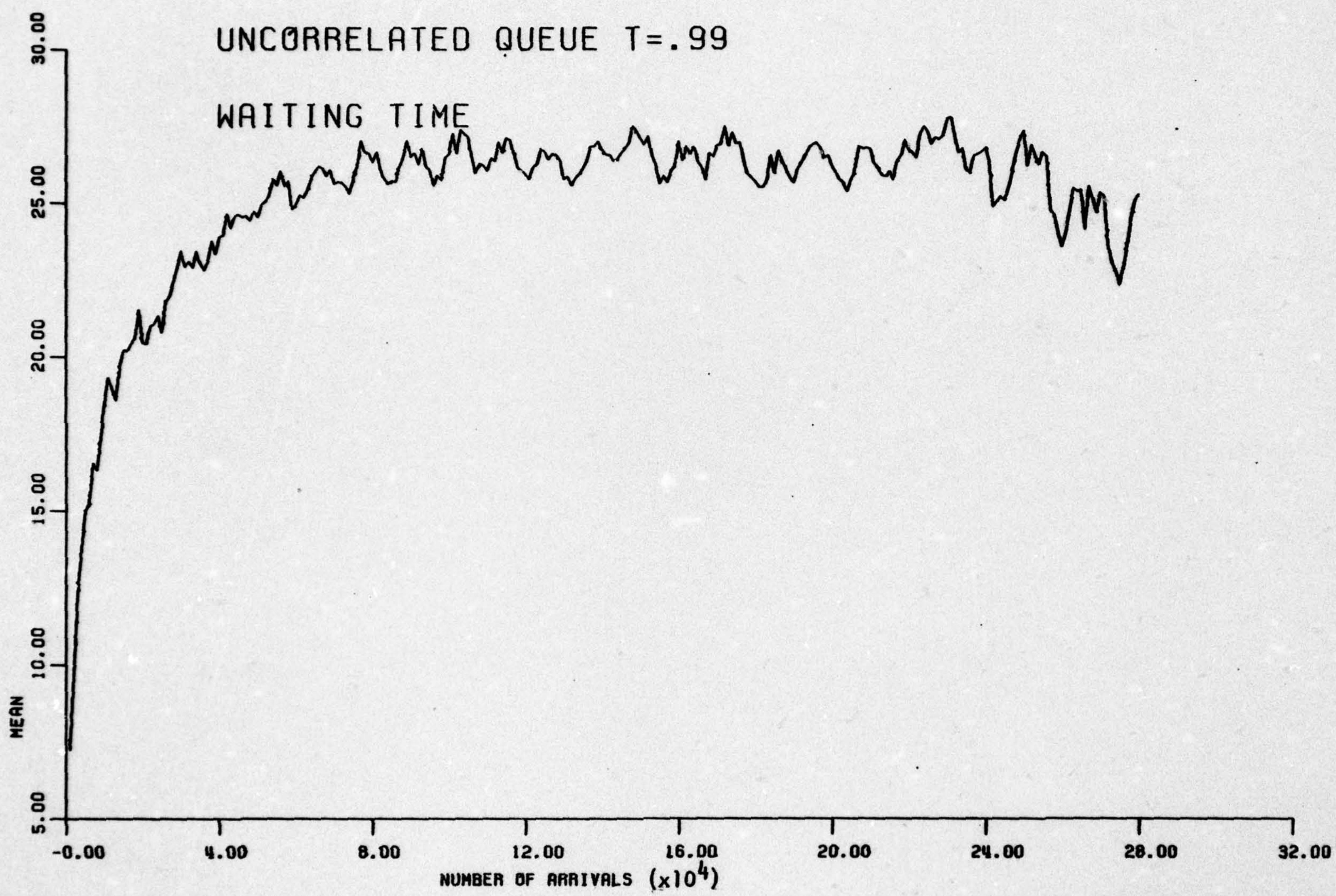

Figure 12c. Estimate $\hat{W}_n$ across 500 replications of the mean waiting time in the M/M/1 queue for $t = 0.99$, and mean service time $E(S) = 0.25$. The mean appears to fluctuate around the known mean (24.75) after 240,000 arrivals.

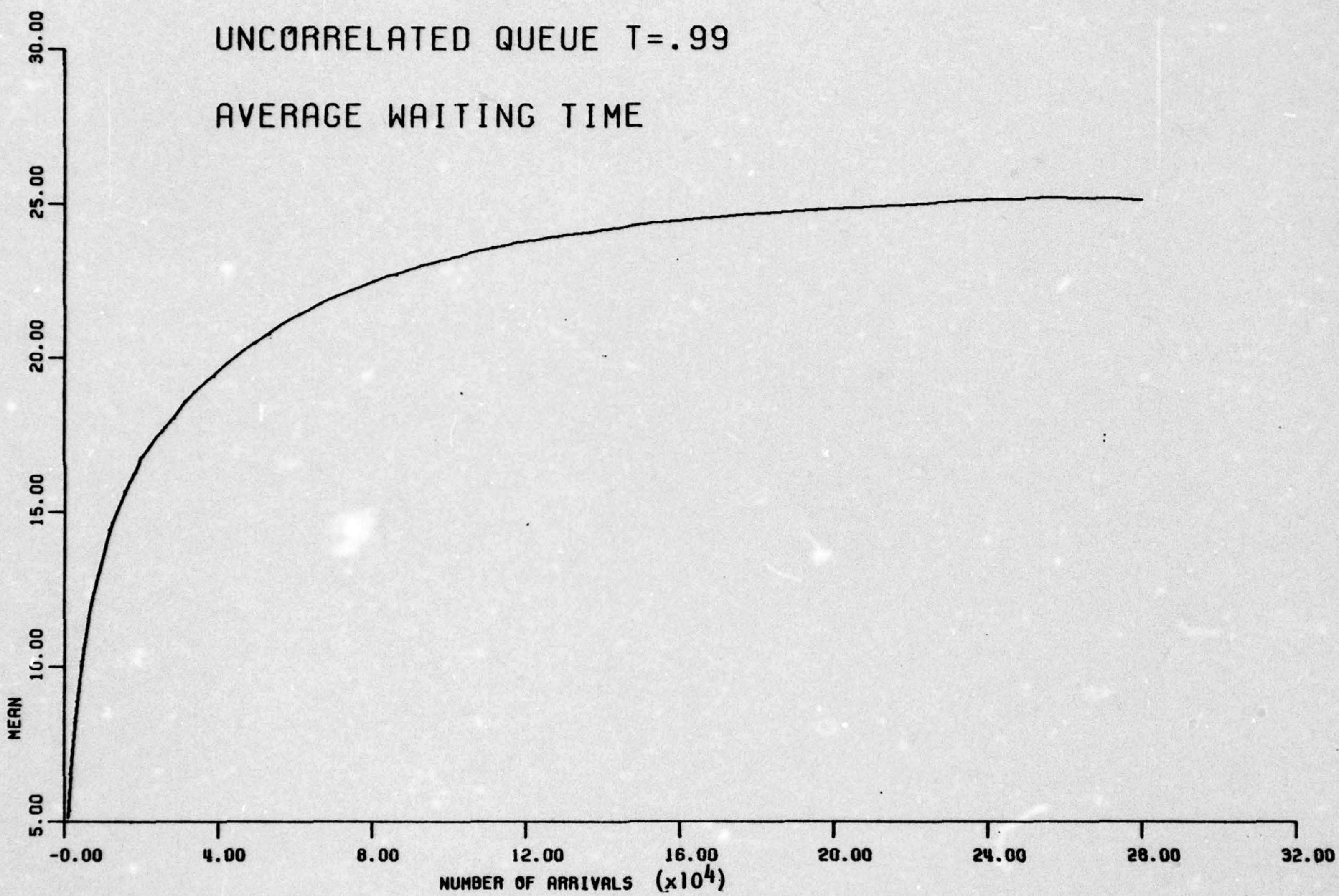

Figure 12d. Estimate $\overline{\overline{W}}_n$ of the stationary mean waiting time in the M/M/1 queue for t = 0.99, and mean service time E(S) = 0.25. This estimate has smaller sampling variance than $\hat{W}_n$ in Figure 12c because it is an average across 500 replications and along 500 sample paths. The convergence is slower than for $\hat{W}_n$ because all $W_n(j)$ including the initial transient, are used.

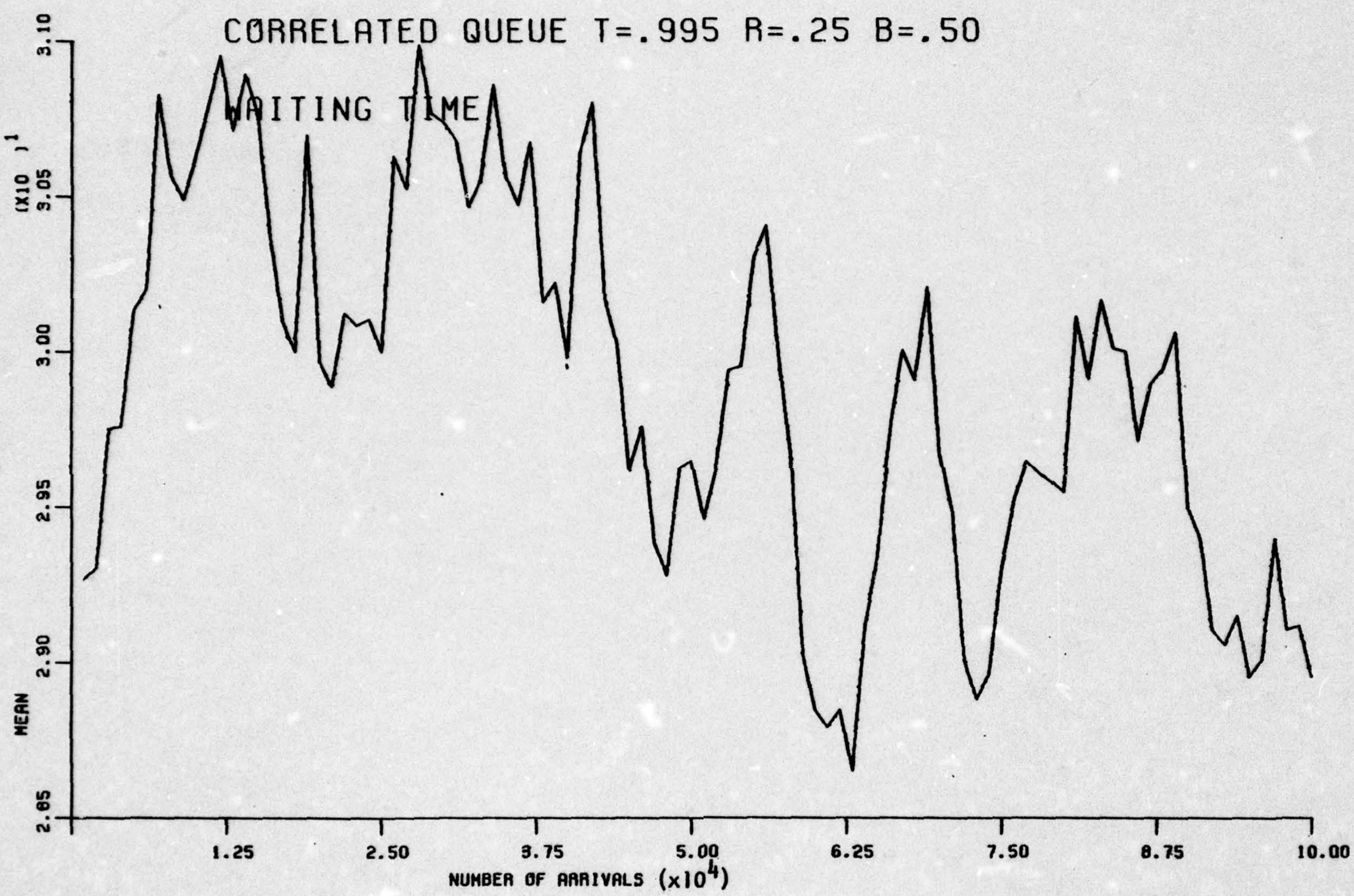

Figure 13a. Estimate $\hat{W}_n$ across 500 replications of the mean waiting time in the correlated queue for $t = 0.995$, $\rho = 0.25$, $\beta = 0.50$. The mean appears to fluctuate around the mean value (29.0) computed by the approximate formula (V.1) after $n = 60{,}000$. To reduce the effect of the initial transient, $W_0(j)$ was taken to be exponential with mean 29.0.

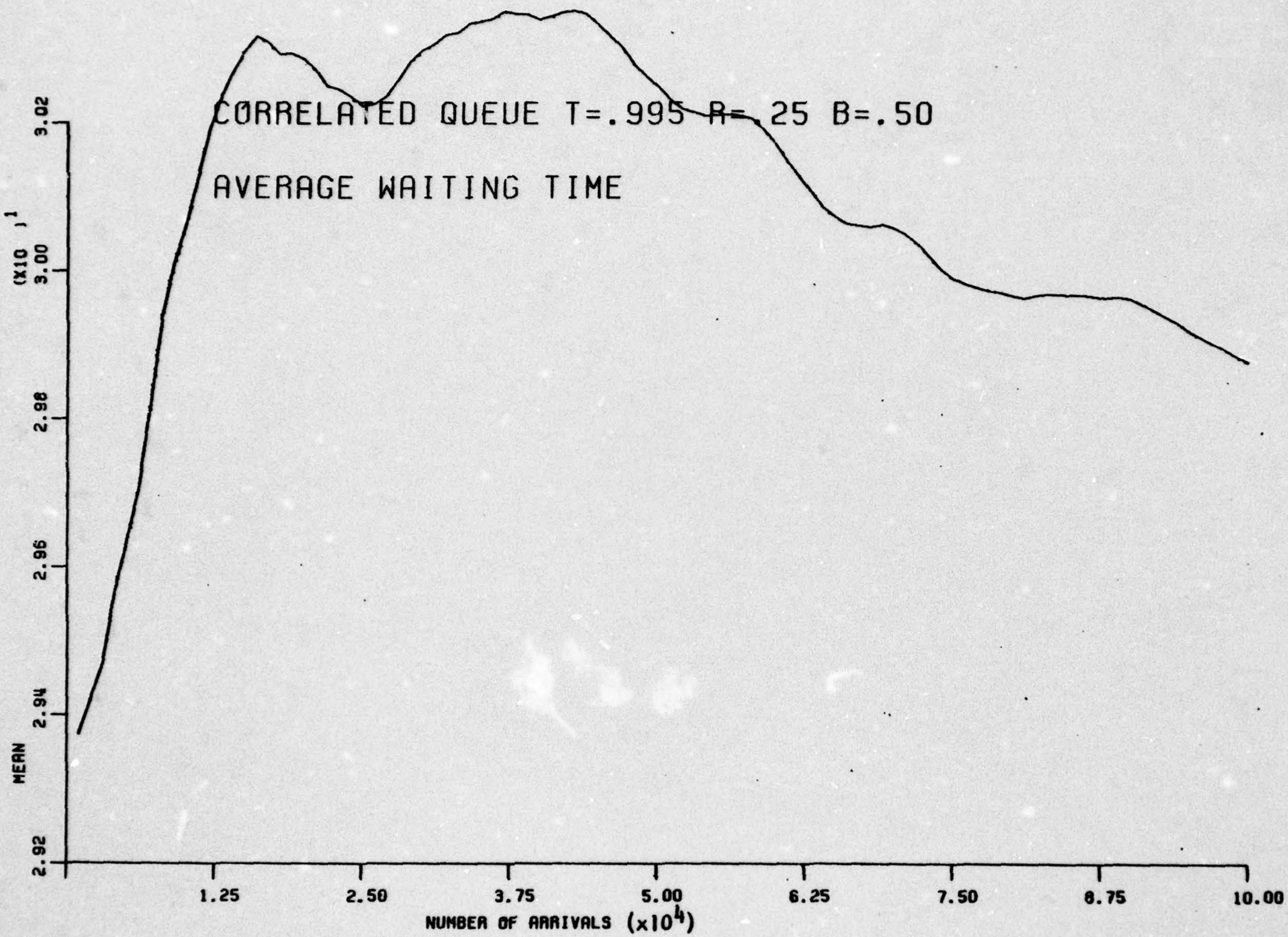

Figure 13b. Estimate $\bar{\bar{W}}_n$ of the stationary mean waiting time in the correlated queue for $t = 0.995$, $\rho = 0.25$, $\beta = 0.50$. This estimate has smaller sampling variance than $\hat{W}_n$ in Figure 13a because it is an average across 500 replications and along the 500 sample paths. The convergence is still slower than for $\bar{W}_n$ because the effect of the initial value, $W_0(j)$ which is not the exact steady state waiting time.

VI. VARIANCE REDUCTION IN THE SIMULATION

A. THE MULTIPLE CONTROL VARIATES METHOD

Suppose we want to estimate the mean, μ_x of a random variable X and suppose V is a zero-mean random variable, or can be made so by subtracting from V its known mean.

Then

$$Y = X - \alpha V \tag{VI.1}$$

is an unbiased estimator for μ_x and

$$\sigma_y^2 = \sigma_x^2 + \alpha^2\sigma_v^2 - 2\alpha \operatorname{cov}[X,V] , \tag{VI.2}$$

where σ_x^2 is the variance of X, σ_v^2 is the variance of V. In order for σ_y^2 to be a minimum given σ_x^2, σ_v^2 and cov(X,V], the optimal choice for α is readily found to be

$$\alpha^* = \frac{\operatorname{Cov}[X,V]}{\sigma_v^2} , \tag{VI.3}$$

and then, substituting back in (V.III), the minimum variance is

$$\sigma_y^2 = \sigma_x^2 - \frac{\operatorname{Cov}^2[X,V]}{\sigma_v^2} = \sigma_x^2(1 - r_{x,v}^2)$$

where $r_{x,v}$ is the coefficient of linear correlation between X and V. Thus the variance of Y, as an estimator of μ_x, is always less than that of X if $|cov(X,V)| > 1$. In a simulation X would be the usual estimator of μ_x, and Y would be some random variable generated in the simulation which is correlated with x.

Generally, COV (X,V) and σ_v^2 are unknown, and we can use in its stead an estimate derived from the simulation. This idea of reducing the variance of an estimator with a control variable is readily extended to a vector V of k control variates $V_1, \ldots, V_k$ with covariance matrix Q. Now let

$$Y = X - \alpha' V$$

where $\alpha' = (\alpha_1, \ldots, \alpha_k)$ is a row vector of coefficients, and standard results from multivariate analysis show that

$$\sigma_Y^2 = \sigma_x^2 + \alpha' Q \alpha - 2\alpha' R .$$

Here, $R_j = cov[X, V_j]$. The optimal choice for α is

$$\alpha^* = Q^{-1} R$$

(assuming, of course that Q is non-singular, for otherwise some control variates would be redundant). Then the minimal value of σ_Y^2 is given by

$$\sigma_Y^2 = \sigma_X^2 - R'Q^{-1}R .$$

Let M be the joint covariance matrix of the control variates, V and X, i.e.,

$$M = \begin{pmatrix} \sigma_x^2 & R' \\ R & Q \end{pmatrix} .$$

Then it can be seen that with the optimal

$$\sigma_y^2 = \frac{\det M}{\det Q} .$$

For further details see Beja (1968) and Kleijnen (1974).

B. SELECTING THE CONTROL VARIABLES

In order to simulate the mean waiting time in the MD x MD/1 queue, several control variables derived from the known properties of the M/M/1 queue and the service and interarrival processes for the MD x MD/1 queue were examined and combined to be the multiple control variables. Note that in the simulation the M/M/1 queue and the MD x MD/1 queue are run with common exponential variates, so that the service and interarrival times in the M/M/1 queue are the variables from which the EARMA type service and interarrival times in the MD x MD/1 queue are generated. It is well known that in a single-server FIFO queue the difference between the cumulative interarrival

times and cumulative service times is a good control for $\bar{W}_n$. The additional control variables were meant to give any greater variance reduction. In the simulation run with $\rho = 0.25$, $t = 0.50$ and $\beta = 0.50$, subroutine CONVAR was used to compute the correlation between waiting time W_n, average waiting time $\bar{W}_n$ in the MD x MD/1 queue and several quantities with known means from uncorrelated queue, i.e., the M/M/1 waiting time, the M/M/1 average waiting time, the number of arrivals since the last regeneration point ($W_n = 0$), sum of service time, etc. The controls for W_n and $\bar{W}_n$ are quite different and are examined separately.

1. Controls for the Waiting Time W_n

The control variables from the uncorrelated queue which have been selected as being the most correlated to W_n are the number of arrivals since the last regeneration (V_1), the waiting time (V_2), the average of the previous 100 interarrival times (V_3), the average of the previous 100 interarrival times (V_4). The correlation matrix for these control variables with W_n is shown in table 5.

Since the mean of these four variables are not zero, the known expected value must be subtracted from each and the estimator is

$$Y = X - \alpha'(V - E(V)) ,$$

where

$$V = (V_1, V_2, V_3, V_4)$$

$$\alpha^* = Q^{-1}R$$

where

$$R = \text{cov}[X,V]$$

and

$$X = \text{waiting time } W_n \text{ (correlated queue).}$$

The values of $E(V)$ used, are as follow:

$$E(V_1) = \frac{t}{(1-t)^2}\ ^*$$

$$E(V_2) = \frac{t}{1-t} \cdot \mu_s$$

$$E(V_3) = \mu_x$$

$$E(V_4) = \mu_s \quad .$$

Note that even though V_4, the sum of the waiting times in the MD x MD/1 queue is an average of correlated random

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*See Appendix I.

variables, the variables are exponential and their average is still μ_s.

A controlled estimate for W_n was obtained using V_1, V_2, V_3, V_4 and the results are shown in Table 6 and Figure 14. The 500 replications makes it possible to estimate, as in Table 5, the values of the components in α^*. In Figure 14 the second, fourth, ... box plots are for the sample of 500 controlled $W_n(j)$"'s, j = 1, ..., 500. There is clearly less variability than in the box plots (first, third, etc.) for the uncontrolled values $W_n(j)$, j = 1, ..., 500. Successive pairs of box plots are for different values of n.

Table 6 gives, for n = 61,000 (1,000) 70,000 statistics of the controlled and uncontrolled $W_n(j)$, j = 1, ..., 500 samples. The values S(MEAN) are to be compared. Thus for n = 70,000, S(MEAN), the estimated standard deviation of $\hat{W}_n$, is 0.06375, which is to be compared to the value 0.04632 for the controlled sample average. Thus the ratio of the standard deviations is 0.04632/0.06375 = 0.726, and the variance reduction is $(0.726)^2 = 0.528$. This is not as much as might have been expected from the multiple controls, but it reflects the difficulty of getting further control variables which are not themselves highly correlated with the previous control variables.

2. Controls for the Sample-path Average $\overline{W}_n$

Different control variables are needed for $\overline{W}_n$ and W_n since the former includes all waiting times up to n and

the latter is local. Thus W_n will only be correlated with local controls, while $\overline{W}_n$ will be highly correlated with controls which contain the whole history of the sample path. Thus 5 control variables from the M/M/1 queue were selected to control $\overline{W}_n$ time, as follows:

V_1: number of zero waiting times up until n in the M/M/1 queue;

V_2: cumulative interarrival time in the M/M/1 queue;

V_3: cumulative service time in the correlated queue;

V_4: cumulative servie time in the M/M/1 queue;

V_5: mean waiting time, $\overline{W}_n$ from the M/M/1 queue.

The estimated matrix from a simulation run with 500 replications at $\rho = 0.25$, $\beta = 0.50$, $t = 0.50$ are shown in Table 7.

The expected values of V which are used to center V are as follows:

$E(V_1) = n(1-t)$, where n = no. of arrivals;

$E(V_2) = \sum_{i=1}^{n} E(X_i)$, where $E(X_i)$ = mean interarrival time of arrivals in the correlated queue;

$E(V_3) = \sum_{i=1}^{n} E(SC_i) = n\mu_S$, where $E(SC_i)$ = mean service time of arrivals in the MD x MD/1 queue;

$$E(V_4) = \sum_{i=1}^{n} E(S_i) = n\ {}_S\ , \qquad \text{where } E(S_i) = \text{mean service time of arrivals in the M/M/1 queue;}$$

$$E(V_5) = \frac{t}{1-t}\,\mu_S\ , \qquad \text{where } \mu_S = \text{mean service time.}$$

The results of the simulation run with 500 replications and $\rho = 0.75$, $t = 0.75$, $\beta = 0.50$ are shown in Table 8 and Fig. 15. From Table 6 we see that, for n = 70,000, S(MEAN) for $\bar{\bar{W}}_n$ with and without control is, respectively 0.001128 and 0.001709 and we recall that S(MEAN) is the estimate of the standard deviation of $\bar{\bar{W}}_n$. These values are of course much smaller than the corresponding values for $\hat{W}_n$. Further the ratio of the estimated standard deviations is 0.001128/0.001709 = 0.660035 and the variance reduction is $(0.660)^2 = 0.4356$, a rather healthy value. The variance reduction shows up dramatically in Figure 15, which is the analog of Figure 14. It is also clear that the controlled $\bar{W}_n$'s are symmetric, possibly normally distributed, and that they have reached steady state.

Control Variable	W_n	V_1	V_2	V_3	V_4
W_n	1.0	0.1679	0.4518	-0.1195	0.1450
V_1	0.1679	1.0	0.6657	-0.2611	-0.3270
V_2	0.4518	0.6657	1.0	-0.2452	0.0330
V_3	-0.1195	-0.2611	-0.2452	1.0	0.3937
V_4	0.1450	-0. 327	0.0330	0.3937	1.0

Control Variables

V_1: Number of arrivals since last regeneration in M/M/1 queue;

V_2: Waiting time of M/M/1 queue;

V_3: Average of 100 previous interarrival times;

V_4: Average of 100 previous service times in MDxMD/1 queue.

Table 5. Correlation Matrix of Waiting Time W_n in the MDxMD/1 Queue and the Control Variables; ρ = .25, t = .50 and β = 0.50.

CORRELATED QUEUE T = C.75 , B = 0.5 , R = 0.75
WAITING TIME

	N = 61000	N = 62000	N = 63000	N = 64CC0	N = 65CCC	N = 66000	N = 67000	N = 68CCC	N = 69C00	N = 70000
MEAN	C.566665	0.882962	0.923667	0.98025C	0.929412	0.887115	C.911407	0.964C11	0.849282	0.958031
S(MEAN)	0.C59753	C.051966	0.055114	0.061826	0.053748	C.C52847	0.055288	0.058342	0.C52331	0.063747
STC. DEV.	1.337CC1	1.161992	1.232395	1.382477	1.201838	1.1817C5	1.236272	1.3C457C	1.17C164	1.425416
SKEWNESS	2.256457	1.967861	1.923409	2.275890	2.083628	2.147532	2.173721	2.173743	2.225035	2.451805
KURTOSIS	6.536824	4.445006	4.196395	5.851847	5.876243	5.865164	6.67487C	5.812078	5.54C057	8.053091
SAMPLE SIZE	500.000000	5CC.C00000	500.000000	500.0000CC	500.000000	500.000C00	500.000000	500.000000	500.00000C	500.000000

CORRELATED QUEUE T = 0.75 , B = C.5 , R = C.75
WAITING TIME (CONTROLLED)

	N = 610CC	N = 62000	N = 63000	N = 64CC0	N = 65C00	N = 66000	N = 67000	N = 6800C	N = 69000	N = 70000
MEAN	0.553056	C.864117	0.964710	0.57C887	0.912510	C.520837	C.922046	C.854934	0.856343	0.920087
S(MEAN)	0.043812	C.C40626	0.041771	0.047355	0.041537	C.C42674	0.043569	0.044033	0.038555	0.046317
STC. DEV.	0.979673	0.908422	0.934026	1.055753	0.928790	0.554222	0.574224	0.584613	0.871C67	1.035676
SKEWNESS	1.305395	1.140954	0.968489	1.321351	1.179124	1.377727	1.4C7943	1.476725	1.397364	1.340769
KURTOSIS	3.143826	3.179940	2.198251	3.223845	3.328954	3.427310	4.6C6538	5.224545	4.125864	3.905669
SAMPLE SIZE	500.00000C	500.000C0C	500.000000	500.000000	500.0C0CCC	500.000000	500.0000C0	500.000000	500.000000	500.000000

Table 6. Estimated Sample Characteristics of Waiting Time Samples, $W_n(j)$, j=1,...,500 With and Without Controls, at 10 Different Values of n.

Control Variable	$\overline{W}_n$	V_1	V_2	V_3	V_4	V_5
$\overline{W}_n$	1.0	-0.4368	-0.1150	0.4719	0.3035	0.4937
V_1	-0.4368	1.0	0.6001	-0.0092	-0.5514	-0.7481
V_2	-0.1150	0.6001	1.0	0.4612	0.0065	-0.4351
V_3	0.4719	-0.0092	0.4612	1.0	0.4872	0.0561
V_4	0.3035	-0.5514	0.0065	0.4872	1.0	0.5248
V_5	0.4937	-0.7481	-0.4351	0.0561	0.5284	1.0

Table 7. Correlation Matrix of Mean Waiting Time $\overline{W}_n$ in the MDxMD/1 Queue and the Five Control Variables; $\rho = 0.25$, $t = 0.50$, $\beta = 0.50$.

CORRELATED QUEUE T = 0.75 , B = 0.5 , R = 0.75
AVERAGE WAITING TIME

	N = 61000	N = 62000	N = 63000	N = 64000	N = 65000	N = 66000	N = 67000	N = 68000	N = 69000	N = 70000
MEAN	0.926575	0.926740	0.926664	0.926622	0.926708	0.926621	0.926644	0.926253	0.926755	0.926665
S(MEAN)	0.001871	0.001853	0.001832	0.001815	0.001812	0.001780	0.001753	0.001728	0.001708	0.001709
STD. DEV.	0.041847	0.041425	0.040571	0.040682	0.040516	0.039805	0.039204	0.038647	0.038198	0.038212
SKEWNESS	0.308746	0.358177	0.322273	0.301338	0.299431	0.271610	0.223433	0.212725	0.202878	0.160221
KURTOSIS	0.372478	0.442900	0.359103	0.323280	0.397227	0.354241	0.302655	0.286622	0.333302	0.280103
SAMPLE SIZE	500.000000	500.000000	500.000000	500.000000	500.000000	500.000000	500.000000	500.000000	500.000000	500.000000

CORRELATED QUEUE T = 0.75 , B = 0.5 , R = 0.75
AVERAGE WAITING TIME (CONTROLLED)

	N = 61000	N = 62000	N = 63000	N = 64000	N = 65000	N = 66000	N = 67000	N = 68000	N = 69000	N = 70000
MEAN	0.925917	0.925998	0.926074	0.925970	0.925927	0.925619	0.925689	0.925604	0.925882	0.925876
S(MEAN)	0.001155	0.001182	0.001182	0.001179	0.001165	0.001154	0.001146	0.001144	0.001136	0.001128
STD. DEV.	0.026721	0.026433	0.026433	0.026363	0.026052	0.025802	0.025629	0.025590	0.025411	0.025228
SKEWNESS	0.359553	0.379427	0.371241	0.358102	0.347002	0.317108	0.268360	0.267535	0.281803	0.275536
KURTOSIS	0.473793	0.459772	0.417157	0.353198	0.325827	0.226232	0.172447	0.230028	0.205625	0.143816
SAMPLE SIZE	500.000000	500.000000	500.000000	500.000000	500.000000	500.000000	500.000000	500.000000	500.000000	500.000000

Table 8. Estimated Sample Characteristics of Estimated Waiting Times, $\overline{W}_n(j)$, j=1,...,500, With and Without Controls at 10 Different Values of n.

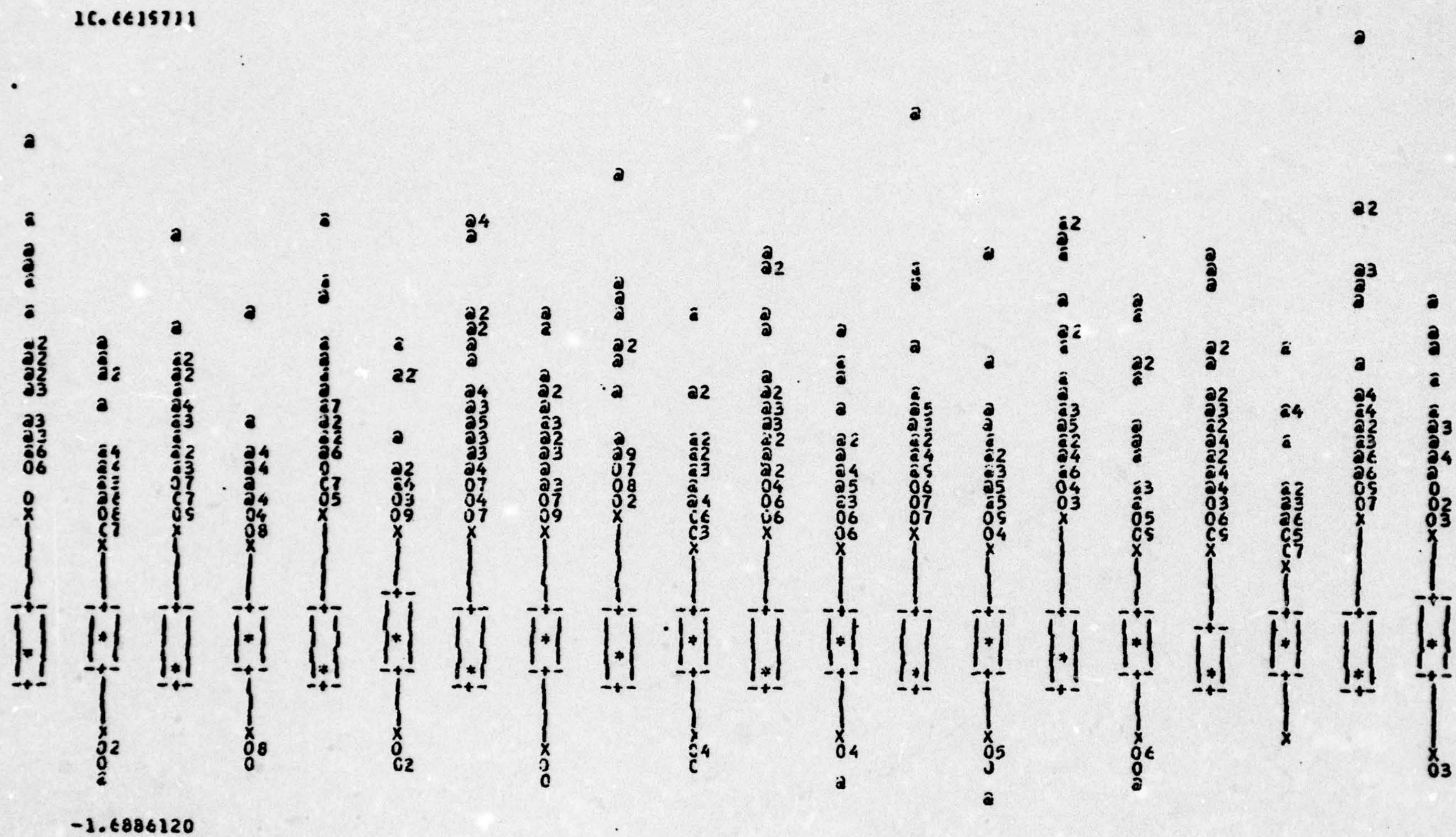

Figure 14. Box plots of samples, with and without controls, for estimating the mean waiting time in a correlated queue with $t = 0.75$, $\beta = 0.5$, $\rho = 0.75$. The first, third,...,nineteenth box plots are of $W_{n_i}(j)$, $j=1,\ldots,500$; for $n_1=61,000$, $n_2=62,000$, ... , $n_{10}=70,000$, respectively. The second, fourth,...,twentieth box plots are of the $W_{n_i}(j)$ with a multiple control. Note that the controlled $W_{n_i}(j)$ may be negative.

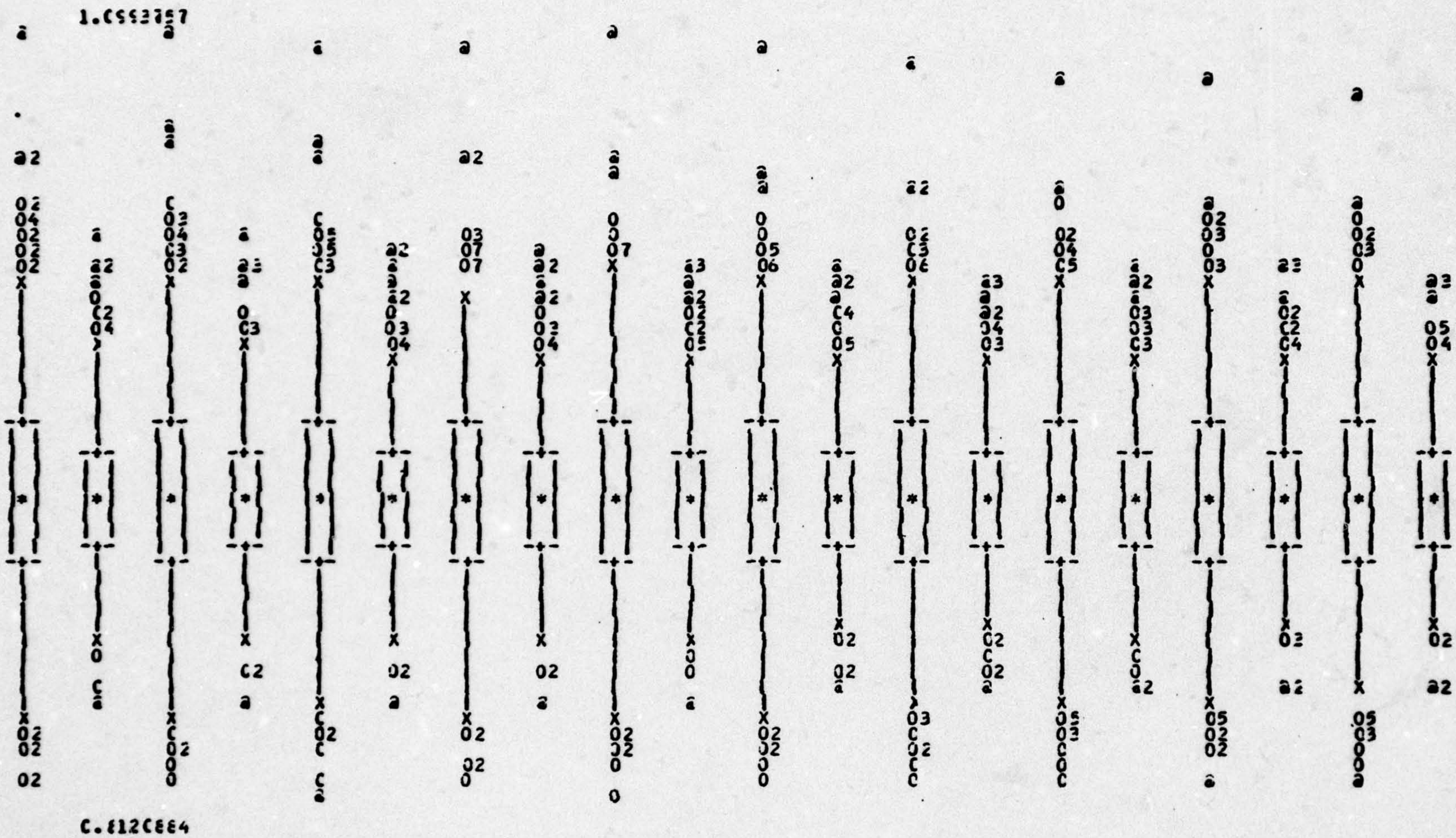

Figure 15. Box plots of samples, with and without controls, for estimating the mean waiting time in a correlated queue with $t = 0.75$, $\beta = 0.5$, $\rho = 0.75$. The first, third, ..., nineteenth box plots are of $\overline{W}_{n_i}(j)$, $j=1,\ldots,500$; for $n_1=61{,}000$, $n_2=62{,}000$, ... , $n_{10}=70{,}000$, respectively. The second, fourth, ..., twentieth box plots are of the $\overline{W}_{n_i}(j)$ with a multiple control.

VII. CONCLUSION

A. In simulation the queue, the box plots, of the sample of waiting times, $W_n(j)$, and average waiting time, $\overline{W}_n(j)$, are helpful in exploring the convergence to steady state. Furthermore the box plots of the controlled waiting times, compared with the uncontrolled waiting times, are useful when examining just how much variance reduction is obtained. Note that the variance reduction attained for $\overline{\overline{W}}_n$ is equivalent to cutting down the sample size by at least a half.

B. An approximation for the mean waiting time for MD x MD/1 queue in the case of $\beta = 0.50$ and $t \to 1$ is found to be equal to $(0.5 + \frac{0.25\rho}{1-\rho})$ times the mean waiting time of M/M/1 queue. Thus the approximate waiting time is

$$E(W) = \frac{t}{1-t}\,\mu_s\,(0.5 + \frac{0.25\rho}{1-\rho}) .$$

C. The multiple control variate technique is useful in reducing the variance of the waiting time and mean waiting time when simulating the correlation queue. This is done by simulating it in parallel with the uncorrelated queue and then using the data from the uncorrelated queue to control the correlated queue. We found that the effectiveness of the variance reduction scheme is better for $\overline{W}_n$, the sample path mean waiting time, than for the waiting time $\hat{W}_n$.

APPENDIX I

Derivation of Mean Time to the Last Regeneration Point (Zero Waiting Time) in an M/M/1 Queue

Let X_i = the number of customers served in the i^{th} busy period of an M/M/1 queue, so the X_i are i.i.d. with some distributions $p_k = P(X_i = k)$; $k \geq 1$, $E(X_i) = \mu$, $Var(X_i) = \sigma^2$, $i = 1,2, \ldots$.

Let $N = \sum_{i=1}^{r} X_i$ be the number of customers in the first r busy periods

and let N_j be the number of busy periods of length j in the first r busy periods, so that

$$\sum_{j=1}^{r} j\, N_j = \sum_{i=1}^{r} X_i \equiv N .$$

Now select a customer at random from the first N, let E_j be the event that the customers are in a busy period of length j, and let S be the number of customers preceding the selected customer in his busy period. Then

1) $P(E_j) = j\, N_j/N$

2) $P(S = k|E_j) = \frac{1}{j}$ for $k = 0,1, \ldots , j-1$

Hence $E(S|E_j) = \frac{1}{j} \sum_{k=0}^{j-1} k = \frac{j-1}{2}$, and

$$E(S) = \sum_{j=1}^{r} \frac{j-1}{2} \cdot j\, N_j/N$$

Since

$$\lim_{r\to\infty} N_j/r = p_j$$

and,

$$\lim_{r\to\infty} N/r = \mu \quad \text{with probability 1,}$$

then

$$\lim_{r\to\infty} E(S) = \sum_{j=1}^{r} \frac{j(j-1)}{2} \frac{p_j}{\mu}$$

$$= \frac{1}{2\mu} \sum_{j=1}^{r} (j^2 p_j - j\, p_j)$$

$$= \frac{1}{2} \left(\frac{\sigma^2 + \mu^2 - \mu}{\mu}\right)$$

$$= \frac{1}{2} \left(\mu + \frac{\sigma^2}{\mu} - 1\right) \quad \text{w.p 1}$$

Since the mean number of customers served in a busy period, μ, is $\frac{1}{1-t}$ and variance σ^2 is $\frac{t + t^2}{(1-t)^3}$ (Cox & Smith, 1961, p. 158), then

$$\lim_{r\to\infty} E(S) = \frac{1}{2} \left(\frac{1}{1-t} + \frac{(t + t^2)/(1 - t)^3}{1/(1 - t)} - 1\right)$$

$$= \frac{t}{(1 - t)^2} .$$

APPENDIX II

PROGRAM LISTINGS

```
C
C      MAIN1 PROGRAM.
C      THIS PRCGRAM CAN BE LSEC TO SIMULATE
C      CROSS CORRELATED QUEUE , EARMA(1,1).
C      INPUT VALUE :-
C        N       = NUMBER OF ARRIVALS
C        RX      = RATE CF INTERARRIVAL TIME
C        RS      = RATE CF SERVICE TIME
C        BETA    = MCVING AVERAGE PARAMETER
C        RHOS    = AUTOREGRESSIVE CORRELATICN PARAMETER
C        KN      = RUN NUMBER
C        KS      = 0 FOR FIRST RUN
C                = 1 FOR CONSECLENCE RUN (READ CATA FRCM THE FILE)
C        NREP    = NUMBER OF REPLICATIONS
C        NSTEP   = NUMBER OF STEP TO LCCK AT THE CATAS
C        NPAST   = NUMBER OF ARRIVALS AT LAST RUN
C        IEX,IES,IBETA,RHCS = SEED TO GENERATE THE RANCOM GENERATORS
C
      REAL*8        ARIV,SERV,WEAR,ARIVM,SERVM,WMBAR,SUMSPA,SUMXPA,
     *              EWBAR,EWMEAR,T,EC,VEC
      CCMMON        RX,RS,F(100C),EX(100C),ES(1CC1),NSTEP
      CCMMON        /A/T(500,14),EC(11)
      READ  (5,100) N,RX,RS,BETA,RHCS
      READ (5,101) KN,KS,NREP,NSTEP,NPAST
      READ  (5,101) IEX,IES,IBETA,IRHCS
         PREV = 100.0
         KSTEF = N/NSTEP
         SL=1.0/RS
         XL=1.C/RX
C
      DC 30 II=1,KSTEP
         IF(II.GT.1) KS= 0
      NCUM = NPAST + II*NSTEP
      NCUMP = NCUM - NSTEP
         NK = NSTEP + KS
         J=0
10       J=J+1
      CALL EXPCN (IEX,EX,NSTEP)
      CALL EXPCN (IES,ES,NK)
C
      DC 11 I=1,NSTEP
         ES(I)=ES(I)*SL
11       EX(I)=EX(I)*XL
         IF(II.GT.1) GO TO 31
         IF(KS.EC.1) GO TO 15
```

```
C
      READ  (2,102) ARIV,SERV,SNEW,WAIT,WBAR,ALAST,SERVM,
     *             SMNEW,WAITM,WMBAR,TL,TR
      GC TO 12
15       ES(NK)=ES(NK)*SL
         SNEW=ES(NK)
         SMNEW=SNEW
         ARIV=C.0D0
         SERV=0.0D0
         WAIT=C.0
         WBAR=0.0D0
         ALAST=C.0
         SERVM=0.0D0
         WAITM=0.0
         WMBAR=C.0D0
         TL=0.0
         TR=0.0
      GC TO 12
31           ARIV  = T(J, 8)
             SERV  = T(J, 9)
             SNEW  = T(J,12)
             WAIT  = T(J, 1)
             WBAR  = T(J, 6)   * DFLOAT(NCUMF)
             ALAST= T(J,13)
             SERVM= T(J,10)
             SMNEW= T(J,14)
             WAITM= T(J, 3)
             WMBAR= T(J,11)   * DFLOAT(NCUMP)
             TL    = T(J, 2)
             TR    = T(J, 7)
12 CALL  CARMA(BETA,RHOS,IBETA,IRHOS,KS,ALAST)
         NP  =  IFIX(PREV*RX/RS)
         SUMXFA = 0.0D0
         SUMSFA = 0.0D0
C
      DC 21 I=1,NSTEP
         SOLD=SNEW
         WOLD=WAIT
         WMOLD=WAITM
         SMOLD=SMNEW
         ARIV=ARIV+EX(I)
         SNEW=F(I)
         SERV=SERV+SNEW
         ARIVM=ARIV
         SMNEW=ES(I)
         SERVM=SERVM+SMNEW
         WAIT=WOLD+SOLD-EX(I)
```

```
      IF(WAIT.LE.0.0) WAIT=0.0
         TL    =  TL + 1.
         WAITM=WMOLD+SMOLD-EX(I)
      IF(WAITM.GT.0.0) GO TO 20
         TR    =  TR + 1.
         TL    = 0.
         WAITM = 0.0
20       WBAR = WBAR + WAIT
         WMBAR=WMBAR+WAITM
      IF(I.LE.NSTEP-NP) GO TO 21
         SUMXPA = SUMXPA + EX(I)
         SUMSPA = SUMSPA + SOLD
21    CONTINUE
      T(J,1)  =   WAIT
      T(J,2)  =   TL
      T(J,3)  =   WAITM
      T(J,4)  =   SUMXPA/DFLOAT(NP)
      T(J,5)  =   SUMSPA/DFLOAT(NP)
      T(J,6)  =   WBAR/DFLOAT(NCUM)
      T(J,7)  =   TR
      T(J,8)  =   ARIV
      T(J,9)  =   SERV
      T(J,10) =   SERVM
      T(J,11) =   WMBAR/DFLOAT(NCUM)
      T(J,12) =   SNEW
      T(J,13) =   ALAST
      T(J,14) =   SMNEW
      IF(II.EQ.KSTEP) WRITE(3,201) ARIV,SERV,SNEW,WAIT,WBAR,ALAST,SERVM,
     *                SMNEW,WAITM,WMBAR,TL,TR
      IF(J.LT.NREP) GO TO 10
      IF(II.EQ.KSTEP) WRITE(7,202) IEX,IES,IBETA,IRHOS
      WRITE (4,303)   (T(I,1),I=1,NREP)
      WRITE (4,303)   (T(I,6),I=1,NREP)
      WRITE (4,303)   (T(I,3),I=1,NREP)
      WRITE (4,303)   (T(I,11),I=1,NREP)
C
C      COMPUTE MEAN OF EACH CONTROL VARIABLE
C
         EC(2) = RX*RS/(RS-RX)**2
         EC(3) = RX/RS/(RS-RX)
         EC(4) = 1.0D0/RX
         EC(5) = 1.0D0/RS
         EC(7)  =   DFLOAT(NCUM)*(RS-RX)/RS
         EC(8)  =   DFLOAT(NCUM)/RX
         EC(9)  =   DFLOAT(NCUM)/RS
         EC(10) =   EC(9)
         EC(11) =   EC(3)
```

```
      CALL CCNVAR (II)
      WRITE (6,200) KN,NCUM,RX,RS,BETA,RHCS
   30 CCNTINUE
      ENC FILE 4
      END FILE 3
  100 FCRMAT (I10,4F10.2)
  101 FCRMAT (5I10)
  102 FCRMAT (5F16.6,/,F16.6,16X,3F16.6,/,F17.7,50X,F7.0,F6.0)
  200 FCRMAT (//1CX,' RUN NUMBER ',I4,1CX,' N = ',I10,//,10X,'RX = ',
     *       F5.2,5X,'RS = ',F5.2,5X,'BETA = ',F5.2,5X,'RHOS = ',F5.2)
  201 FCRMAT (5F16.6,/,F16.6,16X,3F16.6,/,F17.7,50X,F7.0,F6.0)
  302 FCRMAT (4I10)
  3C3 FCRMAT (5F16.7)
      STOP
      END
```

```
C
C     SUBROUTINE CARMA
C       USED TO COMPUTE THE CCRRELATED SERVICE TIME
C       OF THE CROSS- CORRELATED QUEUE.
C
C
C
      SLBROUTINE CARMA (BETA,RHOS,IBETA,IRHOS,K,ALAST)
      CCMMON         RX,RS,F(100C),EX(100C),ES(1CC1),N
      DIMENSICN UR(1000),UB(1CCO)
      RATIO = RX/RS
      ALAST = EX(1)*RATIO*K+ALAST*RHOS*(1-K)
         CALL RANDOM (IRHOS,LR,N)
      IF(UR(1).GT.RHOS) ALAST=ALAST*K+(1-K)*(ALAST+RATIC*EX(1))
      F(1)=BETA*ES(1)
         CALL RANCOM(IBETA,LB,N)
      IF(UB(1).GT.BETA) F(1)=F(1)+ALAST
      DC 10 I=2,N
         ALAST = RHOS*ALAST
         F(I) = BETA*ES(I)
         IF(UR(I).GT.RHOS) ALAST=ALAST+FATIO*EX(I)
         IF(UB(I).GT.BETA) F(I) =F(I)+ALAST
   10 CCNTINUE
      RETURN
      END
```

```
C
C
C     SUBROUTINE CCNVAR
C       USED TO COMPUTE CCNTRCLLED WAITIAG TIPE,
C       AVERAGE WAITING TIPE CF THE QUEUES .
C
C
      SUBROUTINE CONVAR (II)
      IMPLICIT REAL*8 (A-G,M,C-X)
      DIMENSION     IL(6),IM(6),S(11),MEAN(11),XC1(4),XC2(5),STD(11),
     *              ALFA1(4),ALFA2(5),COV(11,11),CCR(11,11),X(11),Y(500),
     *              Q1(4,4),Q2(5,5),R1(4),R2(5),M1(5,5),M2(6,6),YNZ(500)
      CCMMON        /A/T(500,14),EC(11)
         N=500
         NX = 11
         NQ1= 4
         NQ2= 5
         NM1= 5
         NM2= 6
      IF(II.GT.1) GO TO 20
C
C      READ LABELS FOR OUTPUT
C
      READ (5,100) (X(I),I=1,NX)
   20    REP=DFLOAT(N)
         NBAR=0
      DO 31 I=1,NX
         SUM=0.0D0
         SUMSQ=0.0D0
      DO 30 J=1,N
   29    SUM = SUM+T(J,I)
   30    SUMSQ=SUMSQ+T(J,I)**2
         MEAN(I)= SUM/REP
         STD(I)=DSQRT((SUMSQ- SUM**2/REP)/(REP-1.0D0))
         S(I)=SUM
   31 CONTINUE
      DO 34 I=1,NX
      DO 33 J=1,NX
         SCP=0.0D0
      DO 32 K=1,N
   32    SCP     =SCP     +T(K,I)*T(K,J)
         COV(I,J)=(SCP     -S(I)*S(J)/REP)/(REP-1.0D0)
         COR(I,J)=COV(I,J)/STD(I)/STD(J)
   33 CONTINUE
   34 CONTINUE
C
C      PRINT OUT COVARIATE MATRIX.
C
      WRITE (6,200) (X(I),I=1,NX)
      DO 90 I=1,NX
   90 WRITE (6,201) X(I),(COV(I,J),J=1,NX)
```

```
C
C     FRINT CUT CORRELATION MATRIX.
C
      WRITE(6,202) (X(I),I=1,NX)
      DC 91 I=1,NX
   91 WRITE (6,201) X(I),(CCR(I,J),J=1,NX)
      WRITE(6,203) (MEAN(I),I=1,NX)
      WRITE(6,204) (STD(I),I=1,NX)
C
C     COMPUTE THE CONTROL VALUE.
C
      CALL COPYSM (COV,M1,NX,1,NM1,NM1)
      CALL COPYSM (COV,M2,NX,6,NX ,NM2)
      CALL COPYSM (M1,Q1,NM1,2,NM1,NQ1)
      CALL COPYSM (M2,Q2,NM2,2,NM2,NQ2)
      CALL COPYMV (M1,R1,NM1,1,2,NM1,NQ1)
      CALL COPYMV (M2,R2,NM2,1,2,NM2,NQ2)
      CALL DMINV(Q1,NQ1,DQ1,IL,IM)
      CALL DMINV(Q2,NQ2,DQ2,IL,IM)
      CALL DMINV(M1,NM1,DM1,IL,IM)
      CALL DMINV(M2,NM2,DM2,IL,IM)
      CALL MPRCCV (Q1,R1,NQ1,ALFA1)
      CALL MPRCCV (Q2,R2,NQ2,ALFA2)
         VARY1 = DM1/DQ1
         VARY2 = DM2/DQ2
      DC 85 I=1,N
         C1 = 0.0D0
            DC 84 J=2,NM1
               JJ = J-1
   84          C1 = C1 + ALFA1(JJ)*(T(I,J)-EC(J))
               Y(I) = T(I,1)-C1
   85 CCNTINUE
      WRITE (4,230)     (Y(I),I=1,N)
      DC 87 I=1,N
         C2 = 0.0D0
            DO 86 J=7,NX
               JJ = J-6
   86       C2 = C2 + ALFA2(JJ)*(T(I,J)-EC(J))
   87       Y(I) = T(I,6) - C2
      WRITE (4,230)     (Y(I),I=1,N)
      WRITE (6,224) (X(I),I=2,5),(X(I),I=7,11),
     *              (EC(I),I=2,5),(EC(I),I=7,11)
      WRITE (6,222) COV(1,1)
      WRITE (6,220) VARY1
      WRITE (6,223) COV(6,6)
      WRITE (6,221) VARY2
```

```
1CO FCRMAT (10A8)
200 FCRMAT ('1',56X,'COVARIANCE MATRIX',////,13X,11(2X,A8))
201 FCRMAT (/,5X,A8,11F10.4)
202 FCRMAT(///,56X,'CCRRELATION MATRIX',////,13X,11(2X,A8))
203 FCRMAT (///,5X,'MEAN =   ',11F10.3)
204 FCRMAT (/,5X,' STC =   ',11F10.3)
220 FORMAT (////,15X,' REDUCED VAR. CF WAIT  = ' ,F15.5)
221 FCRMAT (////,15X,' REDUCED VAR. OF WBAR  = ' ,F15.5)
222 FCRMAT (////15X,'      VARIANCE OF WAIT = ',F15.5)
223 FCRMAT (////15X,'      VARIANCE OF WBAR = ',F15.5)
224 FCRMAT ('1',23X,4(2X,A8),10X,5(2X,A8)///,5X,
   *        'COMPUTED MEAN  =  ',4F10.3,10X,5F10.3////)
230 FCRMAT (5F16.7)
    RETURN
    END
```

```
C
C
C     SUBROUTINE COPYSM
C       USED TO  COPY ONE MATRIX TO  ANOTHER MATRIX.
C
C
      SUBROUTINE COPYSM(A,B,NA,NA1,NA2,NB)
      IMPLICIT REAL*8 (A-G,M,O-X)
      DIMENSION A(NA,NA),B(NB,NB)
      DO 10 I=NA1,NA2
         II=I+1-NA1
      DO 10 J=NA1,NA2
         JJ=J+1-NA1
10       B(II,JJ)=A(I,J)
      RETURN
      END
C
C
C     SUBROUTINE COPYMV
C       USED TO COPY ONE VECTOR FROM THE MATRIX
C
C
      SUBROUTINE COPYMV (A,B,NA,NR,NA1,NA2,NB)
      IMPLICIT REAL*8 (A-G,M,O-X)
      DIMENSION A(NA,NA),B(NB)
      DO 10 I=NA1,NA2
         II=I+1-NA1
10       B(II)=A(NR,I)
      RETURN
      END
C
C
C     SUBROUTINE MPROCV
C       USED TO COMPUTE THE PRODUCT OF
C       MATRIX AND VECTOR.
C
C
      SUBROUTINE MPROCV (M,V,NC,MP)
      IMPLICIT REAL*8 (A-G,M,O-X)
      DIMENSION M(ND,ND),V(ND),MP(ND)
      DO 10 I=1,NC
         SUMP=0.0D0
      DO 20 J=1,ND
         SUMP=SUMP+M(I,J)*V(J)
20    CONTINUE
         MP(I) = SUMP
10    CONTINUE
      RETURN
      END
```

```
C
C
C     MAIN2 PROGRAM
C       THIS PROGRAM CAN BE USED TO COMPUTE AND DISPLAY
C       THE COMPARISION OF   THE DISTRIBUTION OF
C       WAITING TIME,AVERAGE WAITING TIME
C       BY MEANS OF THE BOXPLOT.
C
C
      DIMENSION   TEMP(10000),W(500,10),WB(500,10),WMIX(500,20),
     *            WC(500,10),WCB(500,10),W1(5000),W2(5000),W3(5000)
     *              ,XW1(10),XW2(10),XW3(10),XW4(10),XW5(10),XCOR(14)
     *              ,XUN(14)
      COMMON        WM(500,10),WMB(500,10)
      COMMON       /A/N1,N2,N3
      EQUIVALENCE   (W(1),W1(1)),(WM(1),W2(1)),(WC(1),W3(1))
     *              ,(WMIX(1),WM(1))
      KSTEP = 10
      N = 500
      NALL = N*KSTEP
C
      DO 11 J=1,KSTEP
      READ (1,100) (W   (I,J),I=1,N)
      READ (1,100) (WB  (I,J),I=1,N)
      READ (1,100) (WM  (I,J),I=1,N)
      READ (1,100) (WMB (I,J),I=1,N)
      READ (1,100) (WC  (I,J),I=1,N)
      READ (1,100) (WCB (I,J),I=1,N)
   11 CONTINUE
C
C     READ LABELS FOR THE OUTPUT.
C
      READ (5,101)  XW1
      READ (5,101)  XW2
      READ (5,101)  XW3
      READ (5,101)  XW4
      READ (5,101)  XW5
      READ (5,101)  XCOR
      READ (5,101)  XUN
      READ (5,102) N1,N2,N3,RX,RS,KIP
      CALL COMPAR (W  ,TEMP,5000,N,KSTEP,50,-2.,XW1,XCOR)
      CALL COMPAR (W  ,TEMP,5000,N,KSTEP,50,0.0,XW2,XCOR)
      CALL COMPAR (WB ,TEMP,5000,N,KSTEP,50,-2.,XW3,XCOR)
      IF(KIP.EQ.1) GO TO 40
      CALL COMPAR (WM ,TEMP,5000,N,KSTEP,50,-2.,XW1,XUN )
      CALL COMPAR (WM ,TEMP,5000,N,KSTEP,50,0.0,XW2,XUN )
      CALL COMPAR (WMB,TEMP,5000,N,KSTEP,50,-2.,XW3,XUN )
   40 CALL COMPAR (WC ,TEMP,5000,N,KSTEP,50,-2.,XW4,XCOR)
      CALL COMPAR (WCB,TEMP,5000,N,KSTEP,50,-2.,XW5,XCOR)
```

```
      CALL STAT (W   ,N,KSTEF,1,XW1,XCCR)
      CALL STAT (W   ,N,KSTEF,C,XW2,XCOR)
      CALL STAT (WB  ,N,KSTEF,1,XW3,XCCR)
      IF(KIP.EC.1) GO TO 30
      CALL STAT (WM  ,N,KSTEF,1,XW1,XUN )
      CALL STAT (WM  ,N,KSTEF,0,XW2,XUN )
      CALL STAT (WMB,N,KSTEP,1,XW3,XUN )
   30 CALL STAT (WC  ,N,KSTEF,1,XW4,XCOR)
      CALL STAT (WCB,N,KSTEF,1,XW5,XCCR)
          II=0
C
      DC 13 I=1,NALL
          IF(W2(I).LE.0.0) GC TC 13
          II = II+1
          TEMP(II) = W2(I)
   13 CCNTINUE
      CALL HISTG (W2,NALL,0)
      WFITE (6,2C0) XW1
      WFITE (6,2C0) XUN
      CALL HISTG (TEMP,II,0)
      WRITE (6,200) XW2
      WRITE (6,200) XUN
          KK=KSTEF*2
      DC 20 K=1,2
      DC 18 J=1,KSTEF
          J2=J*2
          J1=J2-1
      DC 17 I=1,N
          IF(K.EQ.2) GO TO 15
          WMIX(I,J1)=W  (I,J)
          WMIX(I,J2)=WC (I,J)
          GO TC 17
   15     WMIX(I,J1)=WB (I,J)
          WMIX(I,J2)=WCB(I,J)
   17 CCNTINUE
   18 CCNTINUE
          IF(K.EQ.2) GO TO 21
      CALL CCMPAR (WMIX,TEMP,10000,N,KK,50,-2.,XW1,XCCR)
      WRITE (6,303)
      WRITE (6,302) N1,N2,N3
      WRITE (6,301)
      GC TO 20
   21 CALL COMPAR (WMIX,TEMP,10000,N,KK,50,-2.,XW3,XCOR)
      WRITE (6,304)
      WRITE (6,302) N1,N2,N3
      WFITE (6,301)
   20 CCNTINUE
          II=0
      CC 12 I=1,NALL
```

```
      IF(W1(I).LE.0.0) GO TO 12
      II = II+1
      TEMP(II) = W1(I)
   12 CCNTINUE
      CALL HISTG (W1,NALL,0)
      WRITE (6,200) XW1
      WRITE (6,200) XCOR
      CALL HISTG (TEMP,II,0)
      WFITE (6,200) XW2
      WRITE (6,200) XCOR
      CALL HISTG (W3,NALL,0)
      WRITE (6,200) XW4
      WRITE (6,200) XCOR
      NJ=6
      DC 14 I=1,N
      READ(2,103) (W(I,J),J=1,NJ)
      W(I,7)=(W(I,2)-W(I,1))/FLOAT(N2)-1./RS+1./RX
      W(I,8)=(W(I,5)-W(I,1))/FLOAT(N2)-1./RS+1./RX
   14 CCNTINUE
      READ (5,101)  XCOR
      CALL STAT (W,N,8,1,XW1,XCOR)
  100 FCRMAT (5F16.7)
  101 FCRMAT (14A4)
  102 FCRMAT (3I10,2F10.0,I2)
  103 FCRMAT (3F16.6,32X,/,F16.6,16X,2F16.6,16X,/)
  200 FCRMAT (/30X,14A4)
  301 FCRMAT (/15X,' ALTERNATE WITH THE CCNTRCLLEC VAR. CNE ')
  302 FORMAT (/15X,' N = ',I7,' TO',I7,' AT STEP OF',I5)
  303 FCRMAT (/15X,'COMPARE WAITING TIME ')
  304 FCRMAT (/15X,'COMPARE  AVERAGE WAITING TIME ')
      STOP
      END
```

```
C
C     SUBROUTINE COMPAR
C       USED TO DISPLAY THE ECXPLOT
C       TO CCMPARE THE DISTRIBUTION OF EACH COLUMN
C       OF X  MATRIX.
C     CALLING ARGUMENTS:-
C       X     = MATRIX TO CCMPARE WITH NR ROWS AND NC COLUMNS
C       Y     = DUMMY VECTOR WITH DIMENSICN,N = NR*NC
C       ZM    = MISSING VALUE (NOT INCLUDING IN CCMPUTATICNS)
C       XW1,XW2 = LABELS FCR THE OUTPUTS.
C
      SUBROUTINE COMPAR (X,Y,N,NR,NC,IDF,ZM,XW1,XW2)
      DIMENSION X(NR,NC),Y(N),IY(99),A(30),ID(999)
     *               ,XW1(10),XW2(14)
      CCMMON        /A/N1,N2,N3
      DATA A/'    ','|*|','-+-','-*-','-+-','-*-','| |',
     1'=*=',' | ',' X ','   ',' 0 ',' 02',' 03',' 04',
     2' 05',' 06',' 07',' 08',' 09','   ',' @ ',' @2',
     3' @3',' @4',' @5',' @6',' @7',' @8',' @9'/
      ATOM = C.00001
      NY=0
      DC 1 I=1,NR
      DC 1 J=1,NC
      IF(X(I,J).EQ.ZM) GO TC 1
      NY=NY+1
      Y(NY)=X(I,J)
    1 CCNTINUE
      CALL PXSCRT(Y,1,NY)
      WRITE(6,7) Y(NY)
      XMIN=Y(1)
      IF(Y(NY)-Y(1).GE.ATOM) ATOM = 0.0
      AA=(IDP-1.)/(ATOM+Y(NY)-Y(1))
      B=1.-AA*Y(1)
      DC 5 J=1,NC
      NZ=0
      DC 3 I=1,NR
      IF(X(I,J).EQ.ZM) GC TC 3
      NZ=NZ+1
      Y(NZ)=AA*X(I,J)+B
    3 CCNTINUE
      CALL PXSCRT(Y,1,NZ)
      CALL FILL(Y,NZ,IY,ICP)
      DC 4 K=1,IDP
    4 IC(J+NC*(K-1))=IY(IDP+1-K)
    5 CCNTINUE
```

```
      DC 9 K=1,IDP
      IF(NC.LT.20) GO TO 6
      WRITE (6,3C3) (A(1+ID(J+NC*(K-1))),J=1,NC)
      GC TO 9
    6 WRITE (6,300) (A(1+ID(J+NC*(K-1))),J=1,NC)
    9 CCNTINUE
      WRITE (6,8) XMIN
      IF(NC.GT.10) RETURN
      WRITE (6,200) XW1
      WRITE (6,301) N1,N2,N3
      WRITE (6,302) XW2
    7 FCRMAT ('1',1X,F16.7)
    8 FCRMAT (/2X,F16.7)
  200 FCRMAT (/15X,'CCMPARE',10A4)
  300 FCRMAT(30(5X,A3))
  301 FCRMAT (/15X,' N = ',I7,' TO',I7,' AT STEP CF',I5)
  3C2 FCRMAT (/15X,14A4)
  303 FCRMAT (30(3X,A3))
      RETURN
      END
C
C
C      SUBROUTINE FILL
C        USED CONCURRENTLY WITH SUBROUTINE COMPAR
C        TO CCMPUTE AND DISPLAY THE BOXPLOTS.
C
      SUBROUTINE FILL (X,N,IY,ISIZE)
      DIMENSICN X(N),IY(ISIZE)
      ZL=0.5*(X(INT(0.25*N+0.75))+X(INT(0.25*N+1.)))
      ZM=0.5*(X(INT(0.5 *N+0.75))+X(INT(C.5 *N+1.)))
      ZU=0.5*(X(INT(0.75*N+C.75))+X(INT(0.75*N+1.)))
      D=ZU-ZL
      I1=MAXO(INT(0.5+ZL-1.5*D),1)
      I2=MAXO(INT(0.5+ZL-D),1)
      I3=INT(0.5+ZL)
      I4=INT(0.5+ZM)
      I5=INT(C.5+ZU)
      I6=MINO(INT(0.5+ZU+D),ISIZE)
      I7=MINO(INT(0.5+ZU+1.5*D),ISIZE)
      DC 1 K=1,ISIZE
    1 IY(K)=20
      DC 2 K=I1,I7
    2 IY(K)=10
      DC 3 K=I2,I6
    3 IY(K)=0
      DC 4 J=1,N
      IE=INT(C.5+X(J))
      IF(IE.GE.I1) GO TC 5
```

```
    4 IY(IE)=IY(IE)+1
    5 DC 6 K=1,N
      IE=INT(0.5+X(N+1-K))
      IF(IE.LE.I7) GO TO 7
    6 IY(IE)=IY(IE)+1
    7 DC 8 L=J,N
      IE=INT(C.5+X(L))
      IF(IE.GE.I2) GO TO 9
    8 IY(IE)=IY(IE)+1
    9 CC 10 M=K,N
      IG=INT(C.5+X(N+1-M))
      IF(IG.LE.I6) GO TC 11
   10 IY(IG)=IY(IG)+1
   11 DC 12 J=IE,IG
   12 IY(J)=8
      IY(IE)=9
      IY(IG)=9
      DC 13 J=I3,I5
   13 IY(J)=6
      IY(I3)=C
      IY(I5)=C
      IY(I4)=1
      IY(I3)=IY(I3)+2
      IY(I5)=IY(I5)+4
      DC 14 I=I1,I7
   14 IY(I)=MINO(IY(I),19)
      DC 15 I=1,ISIZE
   15 IY(I)=MINO(IY(I),29)
      RETURN
      END
C
C
C       SUBROUTINE STAT
C         USED TC COMPUTE STATISTICS OF THE DATAS
C         WHICH IS REQUIRED.
C
C
      SUBROUTINE STAT (W,NR,NC,NZ,XW1,XW2)
      REAL * 8         SUM,SUMSC,SUM3,SUM4,SCF,S,REF,COUNT
      DIMENSION        COV(10,10),COR(10,10),W(NR,NC),XM(10),SERR(10),S(10),
     *                 SKEW(10),XKURT(10),XMUP(10),XMDOWN(10),Y(10),
     *                 VAR(10),STD(10),CCUNT(10)
     *                  ,XW1(10),XW2(14)
      CCMMON           /A/N1,N2,N3
      DC 20 J=1,NC
         REP = DFLOAT(NR)
         SUM = 0.0D0
         SUMSC= C.0D0
```

```
      COUNT(J) = 0.0D0
   DC 10 I=1,NR
      IF(NZ.EQ.1) GO TO 5
      IF(W(I,J).LE.0.0) GC TO 10
      COUNT(J) = CCUNT(J) + 1.0D0
 5    SUM = SLM + W(I,J)
      SUMSC= SUMSQ + W(I,J)**2
10 CCNTINUE
      IF(NZ.EQ.0) REP = CCUNT(J)
      XM(J)=SUM/REP
      S(J)=SUM
      VAR(J)=(SUMSQ-SUM**2/REP)/(REP-1.0D0)
      STD(J)=SCRT(VAR(J))
      SERR(J) = SQRT(VAR(J)/SNGL(REP))
20 CCNTINUE
   DC 40 J=1,NC
      REP = DFLOAT(NR)
      SUM3=C.0D0
      SUM4=0.CD0
   DC 30 I=1,NR
      IF(NZ.EQ.0.AND.W(I,J).LE.0.0) GO TO 30
      DEV=W(I,J)-XM(J)
      SUM3=SUM3 + DEV**3
      SUM4=SUM4 + DEV**4
30 CCNTINUE
      IF(NZ.EQ.0) REP = COUNT(J)
      XSUM3=SUM3*REP/((REP-1.0D0)*(REP-2.0D0))
      SKEW(J)=XSLM3/STD(J)**3
      XSUM4=SUM4*((REP-2.0C0)*REP+3.CDC)
      XSUM4=XSLM4/((REP-1.CC0)*(REP-2.CC0)*(REP-3.0C0))
      XSUM4=XSUM4-VAR(J)*VAR(J)*3.*SNGL((REP-1.0C0)*(REP+REP-3.0C0))
 C    /SNGL(REP*(REP-2.0CC)*(REP-3.CCC))
      XKURT(J)=XSUM4/(VAR(J)*VAR(J))-3.
40 CCNTINUE
   DC 70 I=1,NC
      Y(I)=I
      XMUP(I)=XM(I)+SERR(I)
      XMDOWN(I)=XM(I)-SERR(I)
   DC 60 J=1,NC
      REP = DFLOAT(NR)
      SCP=0.0D0
      IF(NZ.EQ.1) GO TO 41
      S(I) = C.0D0
```

```
      S(J) = C.ODO
      NCT = 0
41 DC 50 K=1,NR
      IF(NZ.EQ.1) GO TO 42
      IF(W(K,I).LE.0.0.OF.W(K,J).LE.C.0) GC TC 5C
      S(I) = S(I)+W(K,I)
      IF(I.NE.J) S(J) = S(J)+W(K,J)
      NCT = NCT+1
42    SCP=SCP+W(K,I)*W(K,J)
50 CCNTINUE
      IF(NZ.EC.0) REP=DFLCAT(NCT)
      COV(I,J)=(SCP-S(I)*S(J)/REP)/(REF-1.0CC)
      IF(NZ.EQ.0) GO TO 51
      COR(I,J)=CCV(I,J)/STC(I)/STD(J)
      GO TC 60
51    COR(I,J) = COV(I,J)/SCRT(COV(I,I)*COV(J,J))
60 CCNTINUE
70 CCNTINUE
   WRITE (6,200)
   DC 91 I=1,NC
91 WRITE(6,201) (COR(I,J),J=1,NC)
      IF(NC.EQ.8) GO TO 92
   WRITE (6,2C3) XW1
   WRITE (6,301) N1,N2,N3
   WRITE (6,302) XW2
   WRITE (6,303) XW2,XW1,(I,I=N1,N2,N3)
   GC TO 93
92 WRITE (6,401) XW2,N2
   WRITE (6,402) N2,XW2
   WRITE (6,404) (XM   (I),I=1,NC)
   WRITE (6,405) (SERR (I),I=1,NC)
   WRITE (6,406) (STC  (I),I=1,NC)
   WRITE (6,407) (SKEW (I),I=1,NC)
   WRITE (6,40E) (XKLRT(I),I=1,NC)
   WRITE (6,409) (REP,I=1,NC)
   RETURN
93 WRITE (6,304) (XM   (I),I=1,NC)
   WRITE (6,305) (SERR (I),I=1,NC)
   WRITE (6,306) (STD  (I),I=1,NC)
   WRITE (6,307) (SKEW (I),I=1,NC)
   WRITE (6,308) (XKLRT(I),I=1,NC)
   IF(NZ.EC.0) WRITE (6,3C9) (COUNT(I),I=1,NC)
```

```
      IF(NZ.EC.1) WRITE (6,3C9) (REP      ,I=1,NC)
      DC 94 I=1,NC
      III=N1/1000+I-1
      IF(NZ.EC.0) WRITE(7,5C1)III,XM(I),SERR(I),STD(I),SKEW(I),XKURT(I)
     *,CCUNT(I)
      IF(NZ.EC.1) WRITE(7,501)III,XM(I),SERR(I),STC(I),SKEW(I),XKURT(I)
     *,REP
   94 CCNTINUE
  200 FCRMAT ('1')
  201 FCRMAT (/2CX,10F10.4)
  203 FCRMAT (/30X,'CORRELATICN MATRIX',10A4)
  204 FCRMAT (/15X,'MEAN',1CA4)
  205 FCRMAT (/15X,'SKEWNESS',10A4)
  206 FCRMAT (/15X,'KURTCSIS',10A4)
  301 FCRMAT (/30X,' N = ',I7,' TO',I7,' AT STEP OF',I5)
  302 FCRMAT (/30X,14A4)
  303 FCRMAT ('1',55X,14A4,//,56X,10A4,///,13X,1C(' N =',I7,1X))
  304 FCRMAT (/,' MEAN        ',10F12.6)
  305 FCRMAT (/,' S(MEAN)     ',10F12.6)
  306 FCRMAT (/,' STD. DEV.   ',10F12.6)
  307 FCRMAT (/,' SKEWNESS    ',10F12.6)
  308 FCRMAT (/,' KURTOSIS    ',10F12.6)
  309 FCRMAT (/,' SAMPLE SIZE',10F12.6)
  401 FCRMAT (/26X,A4,2X,2A4,2(6X,A4),4(2X,2A4),A4,//,50X,
     *'CORRELATICN MATRIX AT N = ',I7)
  402 FCRMAT ('1',55X,'SUMMARY STATISTIC AT N = ',I7,///,
     *19X,A4,5X,2A4,2(7X,A4,2X),2X,4(3X,2A4,2)),1X,A4)
  404 FCRMAT (/,' MEAN        ',2F13.6,2F13.7,F13.6,3F13.7)
  405 FCRMAT (/,' S(MEAN)     ',2F13.6,2F13.7,F13.6,3F13.7)
  406 FCRMAT (/,' STD. DEV.   ',2F13.6,2F13.7,F13.6,3F13.7)
  407 FCRMAT (/,' SKEWNESS    ',2F13.6,2F13.7,F13.6,3F13.7)
  408 FCRMAT (/,' KURTOSIS    ',2F13.6,2F13.7,F13.6,3F13.7)
  409 FCRMAT (/,' SAMPLE SIZE',2F13.6,2F13.7,F13.6,3F13.7)
  501 FCRMAT(I4,6F12.6)
      RETURN
      END
```

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